Math 55a: Honors Abstract Algebra
Homework 3

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Fine, I'll give-in to the numbering system this time.

1 Implications of Spanning Set Countability

(a) Prove a vector space over a countable field that has a countable spanning set is countable as well.

Let $V_s = \{v_1, v_2, \ldots\}$ be the countable spanning set and $F = \{a_1, a_2, \ldots\}$ be the countable field. Therefore for each vector v_i in V_s, the set of numbers $a_j v_i$ is countable. In fact it is equivalent to the F itself. Thus the cardinality of the span of V_s can not be more than the cardinality of the union of $|V_s|$ sets each with distinct elements and each of size $|F|$. Such a union would be countable [3], and therefore so is the vector span of which V_s spans.

(b) Prove that any vector space with a countable spanning set over any field does not have an uncountable linearly independent set.

This is already proven to us via Axler's Proposition 3.16 [1, pg 92] of chapter four.

2 Implications of Vector Space Countability on Spanning Sets

3 Problems 6 and 22 of Axler's Third

(a) Problem 6

Let $S_1, S_2 \ldots S_n$ all be injective, linear mappings. Such that $S_1 S_2 \cdots S_n$ is itself a mapping from, say V, to some arbitrary vector space. Therefore, if we have that

$$S_1 \cdots S_n(u) = S_1 \cdots S_n(v)$$

for some vectors $u, v \in V$, then by injective property of S_1 we also have the following.

$$S_2 \cdots S_n(u) = S_2 \cdots S_n(v)$$

Likewise by S_2's property of injection we also have that

$$S_3 \cdots S_n(u) = S_3 \cdots S_n(v)$$

and so on, until the continuation of this pattern arrives at the final implication that $u = v$. Hence $S_1 \cdots S_n$ is an injection.

If $S_1 \cdots S_n$ is injective, what can be said about each individual mapping? Each individual mapping is also injective in this case.

Assume that $S_1 \cdots S_n$ is an injective mapping with some vector space V as its domain. Let $S_1 \cdots S_n(u) = S_1 \cdots S_n(v)$ for some $u, v \in V$. From which we know that $u = v$ by the injective property of $S_1 \cdots S_n$. Thus we have that $S_n(u) = S_n(v)$ is also true, which in turn implies that that $S_1 \cdots S_{n-1}$ is injective. We can continue with this sequence of "if-this-than-that" $n-1$ more times, arriving at the final conclusion that the mappings

$$\begin{gathered} S_1 \\ S_1 S_2 \\ \vdots \\ S_1 \cdots S_n \end{gathered}$$

are all injective.

Now assume that $S_i(u') = S_i(v')$ for $i \in \{1, \ldots, n\}$ and some vectors u', v' in the domain of S_i. Therefore we have that

$$S_1 \cdots S_{i-1} S_i(u') = S_1 \cdots S_{i-1} S_i(v')$$

which, by the above result, implies that $u' = v'$. Hence S_i is an injective mapping, and subsequently, so are all the mappings $S_1, \ldots, S_n$.

(b) Problem 22

Assume that V is a vector space and $S, T \in \mathcal{L}(V)$ are such that ST is invertible. Let $T(u) = T(v)$ for some vectors $u, v \in V$. Therefore, $ST(u) = ST(v)$ which implies that $u = v$ since ST is an injection. Thus T is also an injection. Hence the by Axler's Theorem 3.21 [2, pg 57], T is invertible.

In a similar manner, if we now assume that $S(u) = S(v)$ then there exists a $u', v' \in V$ such that $ST(u') = S(u) = S(v) = ST(v')$, by T's surjectivity. Hence because ST is injective then $u' = v'$, which indicates that $u = v$ since T is a bijection for which $T(u') = u$ and $T(v') = v$ are true. Thus S is an injection, for which Axler's Theorem 3.21 [2, pg 57] yields to us that S is indeed invertible.

For the opposite direction, assume that S and T are each, individually, invertible mappings. From the solution to Axler's problem six in chapter three (above) we learned that ST is therefore an injection. Thus, again using Axler's Theorem 3.21 [2, pg 57], we have that ST too is invertible.

4 Problems 23 and 24 of Axler's Third

(a) Problem 23

Let $S, T \in \mathcal{L}(V)$. Then assuming that we have $ST = I$ then the following equation holds for some $v \in V$.

$$S(v) = IS(v) = (ST)S(v) = S(TS)(v)$$

Hence $TS = I$. The other direction is similarly proven by this proof due to symmetry.[1]

(b) Problem 24

Assume that $T \in \mathcal{L}(V)$ is a scalar multiple of the identity tranformation on $\mathcal{L}(V)$. Let this scalar be a. Hence for all $S \in \mathcal{L}(V)$ the following equation holds for some $v \in V$.

$$ST(v) = S(T(v)) = S(av) = aS(v) = T(S(v)) = TS(v)$$

Therefore we can see that $ST = TS$.

[1]Yikes! Its a little strange (maybe disconcerting too) to me that the problem started with the assumption that V is finite dimensional, but yet my proof here makes no use of that property (Insert quivering-mouthed emoticon here).

5 Polynomials on $\mathbb{R}$ and $\mathbb{C}$ Vector Spaces

6 Evaluation Map

(a) Show that the evaluation map is a linear transformation.

The following demonstrates the additivity of the evalutation map for some vectors $L, S \in \mathcal{L}(V, W)$ and $v \in V$.

$$E_v(L+S) = (L+S)(v) = L(v) + S(v) = E_v(L) + E_v(S)$$

With the following, we have homogeneity for some a in the field over which V and W lie.

$$E_v(aL) = (aL)(v) = a(L(v)) = aE_v(L)$$

Hence the evaluation map is a linear transformation.

7 Linear Maps and $\mathbb{Q}$

Let V and W be vector spaces over the field of rationals, $\mathbb{Q}$, and let T be a map from V to W. Proving additivty of T if T were to be linear is trivial; it's one part of linearity, so we'll simply assume additivity and prove homogeneity to gain linearity.

To aide in our proof let us first prove homogeneity when over Z. So assume that $n \in \mathbb{Z}$. Then we have

$$nT(v) = \overbrace{T(v) + \cdots + T(v)}^{n \text{ times}} = T(\overbrace{v + \cdots + v}^{n \text{ times}}) = T(nv). \tag{7.1}$$

Now let $q = \frac{p}{r} \in \mathbb{Q}$. Using equation 7.1 and a small trick, we get the following.

$$qT(v) = \frac{p}{r}T(v) = \frac{1}{r}\left(pT(v)\right) = \frac{1}{r}T(pv) = \frac{1}{r}T\left(r\frac{p}{r}v\right) = \frac{1}{r}\left(rT\left(\frac{p}{r}v\right)\right) = T\left(\frac{p}{r}v\right) = T(qv)$$

Thus we have that T is linear.

8 Existence of Unique Dual Bases

I accidentally read through the part of the Wikipedia article, [4], that described the basis. Oh well, it's kind of like conversing with a peer about the problem of which she already knows the answer, and she lets it slip.

Anyways, I realized after goin through the problem that if I just would have applied the "Kronecker-delta" result itself, the UNIQUE basis would have directly revealed itself.

Assume that V is a finite-dimensional vector space with $v_1, \ldots, v_n$ as its basis, and that V^* the vector space of linear maps from V to F, where F is the field over which V lies.

The existence and uniqueness of "Kronecker Maps" Let the mapping v_j^* in V^*i for 1 through n be such that, for all v,

$$v_j^*(v) = v_j(c_1 v_1 + \cdots + c_n v_n) = c_j$$

then its easy to see that the j^{th} of these mappings will each for the Kronecker map for each of the vectors in the basis of V mentioned above.

Assume by way of contradiction that for at least one j, v_j^* is not unique, taking $e_j \in V^*$ to be one such mapping with $e_j \neq v_j^*$ but $e_j(v_i) = \delta_{ij}$ for the basis, $v_1, \ldots, v_n$. Therefore we have the following set of equations for each $v \in V$.

$$\begin{aligned} e_j(v) &= e_j(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 e_j(v_1) + \cdots + c_j e_j(v_j) + \cdots + c_n e_n(v_n) \\ &= 0 + \cdots + c_j + \cdots + 0 \\ &= c_j \end{aligned}$$

Hence we have a contradiction and thus, there is no alternative to v_j^*; its unique!

These maps are a basis for the dual space V^*. By the following sequence of equations for an arbitrary vector, e, in the dual space of V, we have that $v_1^*, \ldots, v_n^*$ spans V^*, if we let $e(v_i) = c_i'$ for each i of the indicies of the basis of V.

$$\begin{aligned} e(v) &= e(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 e(v_1) + \cdots + c_n e(v_n) \\ &= v_1^*(v) c_1' + \cdots + v_n^*(v) c_n' \\ &= (c_1' v_1^*)(v) + \cdots + (c_n' v_n^*)(v) \\ &= (c_1' v_1^* + \cdots + c_n' v_n^*)(v) \end{aligned}$$

Assume by way of contradiction that this set of vectors in the dual space is not linearly independent. Thus, at the very least, one of these vectors is a linear combination of the others. Let this said vector be v_i^*, which therefore implies that

$$\begin{aligned} -v_i^*(v) &= (v_1^* + \cdots + v_{i-1}^* + v_{i+1}^* + \cdots + v_n^*)(v) \\ c_i &= -(c_1 + \cdots + c_{i-1} + c_{i+1} + \cdots + c_n) \end{aligned}$$

for some vector $v = c_1 v_1 + \cdots + c_n v_n \in V$. However, this leads to the following set of equations

$$\begin{aligned} v &= c_1 v_1 + \cdots + c_n v_n \\ &= c_1 v_1 + \cdots + c_{i-1} v_{i-1} + (-(c_1 + \cdots + c_{i-1} + c_{i+1} + \cdots + c_n)) v_i + c_{i+1} v_{i+1} \\ &= c_1(v_1 - v_i) + \cdots + c_{i-1}(v_{i-1} - v_i) + c_{i+1}(v_{i+1} - v_i) + \cdots + c_n(v_n - v_i) \end{aligned}$$

in which we see that any vector $v \in V$ is a linear combination of $n-1$ vectors, which considering that V has dimension n is a contradiction. Hence we have that $v_1^*, \ldots, v_n^*$ is a basis for the dual space, V^*.

9 Dual of Direct Sum of Multiple Vector Spaces

What in the world is I?

10 Dual Bases in F^{m+1}

Let $x_0, x_1, \ldots, x_m$ each be distinct elements of F. Let A be $m+1 \times m+1$ matrix whose i^{th} column is the i$^{\text{th}}$vector of the set of vectors, $v_i := (x_0^i, x_1^i, \ldots, x_m^i)$. Since we have seen that this set of vectors forms a basis for F^{m+1}, then we know that the following holds for an arbitrary vector v, and some scalars c_j.

$$v = A(c_0, c_1, \ldots, c_m)^T$$

This in turn yields to us the following.

$$(c_0, c_1, \ldots, c_m)^T = A^{-1}v$$

Since the columns of A form a basis, by construction, then we know that it is invertible and that A^{-1} exists. Hence we can finally see that

$$c_j = e_j A^{-1} v$$

where e_j is simply the j^{th} vector of the standard basis. Thus we are left to concluded that the j^{th} element of the dual basis of F^{m+1} is nothing more than the operation induced by multiplication by $e_j A^{-1}$.

References

[1] Artin, Michael. *Algebra*. Prentice Hall. Upper Saddle River NJ: 1991.

[2] Axler, Sheldon. *Linear Algebra Done Right* 2nd Ed. Springer. New York NY: 1997.

[3] Rudin, Walter. *Mathematical Analysis* 3rd Ed. McGraw-Hill. New York NY: 1976.

[4] "Dual Space", `http://en.wikipedia.org/wiki/Dual_space`.