Math 55a: Honors Abstract Algebra
Homework 4

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Allow V to be an F-vector space with projective space, $\mathbf{P}V$.

(i) Points and lines in $\mathbf{P}V$

— (a) —

Let U_1 and U_2 be distinct "points" of the projective space $\mathbf{P}V$. Because they're distinct, then their intersection is simply the zero vector since they are each one dimensional. This gives us

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = 1 + 1 - 0 = 2.$$

So any subspace W containing both U_1 and U_2 will have dimension of at least 2, since it will necesarily contain $U_1 + U_2$. In particular if W has dimension 2, then W is the space $U_1 + U_2$, and so the unique line containing both U_1 and U_2 is $U_1 + U_2$.

— (b) —

Let U_1 and U_2 be distinct lines both contained in some projective plane, say W. Being lines, $\dim U_1 = \dim U_2 = 2$, but since they are distinct then $\dim(U_1 + U_2) > 2$. Now since $\dim W = 3$, then $\dim(U_1 + U_3) \leq 3$ and therefore $\dim(U_1 + U_2)$ is squeezed to be 3. Thus we have

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) = 2 + 2 - 3 = 1$$

and therfore $U_1 \cap U_2$ is a point.

(ii)

(iii)

2 Axler: Chapter 5, Exercise 4

Problem Statement. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\ker(T - \lambda I)$ is invariant under S for every $\lambda \in F$.

Problem Solution. Let v be in the kernel of $T - \lambda I$. Therefore we have that for any $v \in \ker(T - \lambda I)$,

$$(T - \lambda I)v = 0$$

Applying S to both sides of the above equation yields the following sequence.

$$\begin{aligned} S(T - \lambda I)v &= 0 \\ (ST - \lambda S)v &= 0 \\ (TS - \lambda S)v &= 0 \\ (T - \lambda I)Sv &= 0 \end{aligned}$$

Hence $\ker(T - \lambda I)$ is invariant under the S operator.

3 Axler: Chapter 5, Exercise 7

Problem Statement. Suppose n is a positive integer and $T \in \mathcal{L}(F^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n);$$

in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

Problem Solution. If $(x_1, \ldots, x_n)$ were to be an eigenvector of T, then it must have the property that

$$\lambda(x_1, \ldots, x_n) = \left(\sum_i x_i, \ldots, \sum_i x_i \right)$$

indicating that each x_j is $1/\lambda$ of the sum of all the elements comprising the vector. Hence all eigenvectors of T must be of the form $(x, \ldots, x)$.

4 Axler: Chapter 5, Exercise 8

Problem Statement. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(F^\infty)$ defined by

$$T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$$

Problem Solution. The eigenvectors are such that

$$(\lambda z_1, \lambda z_2, \lambda z_3, \ldots) = (z_2, z_3, \ldots)$$

where λ is a possible eigenvector of T. Thus we have that $z_2 = z_1\lambda,\ z_3 = z_2\lambda,\ z_4 = z_3\lambda, \ldots$; meaning that the i^{th} element of every eigenvector corresponding to an eigenvalue, λ, of T is of the following form.

$$z_i = z_1 \lambda^{i-1}$$

Thus the an eigenvector of T is going to be a vector whose terms are the terms of a geometric progression, and associated eigenvalue is going to be the multiplier (or as wikipedia tells me, the "common ratio") of the said progression.

5 Axler: Chapter 5, Exercise 9

Problem Statement. Suppose that $T \in \mathcal{L}(V)$ and $\dim(T(V)) = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

Problem Solution. Since the dimension of the image of T is k, then there is no linearly independent set of vectors of size larger than k in the image. Since eigenvectors for a given eigenvalue have the form

$$Tv = \lambda v$$

then all vectors in a given set of eigenvectors corresponding to distinct eigenvalues will be in the image of T, and thus limited in size to k, which in turn limits the number of such eigenvectors to k. Those eigenvalues, however, would be non-zero, thus throwing in with them, the eigenvalue zero, we get that the number of distinct eigenvalues is limited to $k + 1$ (note that only non-zero eigenvalue have eigenvectors in the image, and not in the kernel).

6 Axler: Chapter 5, Exercise 10

Problem Statement. Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda \in F \setminus \{0\}$. Provethat λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.

Problem Solution. Let $T \in \mathcal{L}(V)$ be an invertible mapping and, a nonzero, $\lambda \in F$ be an eigenvalue of T. Let v be an eigenvector for corresponding to λ. In this case we have the following.

$$\begin{aligned} Tv &= \lambda v \\ v &= \lambda T^{-1} v \\ \frac{1}{\lambda} v &= T^{-1} v \end{aligned}$$

Therefore we can see that the inverse of T has $1/\lambda$ as an eigenvalue. The proof of the reverse direction is equivalent to the above, making the obvious substitution.

7 Axler: Chapter 5, Exercise 11

Problem Statement. Let $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eignevalues.

Problem Solution. Let λ be an eigenvalue of ST. Then there exists a $v \in V$ such that $STv = \lambda v$. Multiplying both sides by T we get

$$TS(Tv) = T(\lambda v) = \lambda(Tv)$$

and thefore λ is also an eigenvalue for TS.

8 Axler: Chapter 5, Exercise 12

Problem Statement. Suppose that $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is just a scalar multiple of the identity operator.

Problem Solution. ~~I don't know how to prove this for infinite dimensional vector spaces, so assuming finite dimensional, we~~[1] We have that for $v \in V$ with eigenvalue λ where $\dim V = n$ and $\lambda_1, \ldots, \lambda_n$, which may or may not be distinct, are the eigenvalues for some basis $v_1, \ldots, v_n$.

$$\begin{aligned} Tv &= Tv \\ \lambda v &= T(a_1 v_1 + \cdots + a_n v_n) \\ \lambda(a_1 v_1 + \cdots + a_n v_n) &= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \\ a_1 \lambda v_1 + \cdots + a_n \lambda v_n &= a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n \end{aligned}$$

Hence, we can see that every eigenvalue of each vector in the above basis is the same. This implies that every vector in the space has the same eigenvalue, and, calling that value λ, then $T = \lambda I$.

9 Axler: Chapter 5, Exercise 15

Problem Statement. Suppose $F = C$, $T \in \mathcal{L}(V)$, $p \in P(C)$, and $a \in C$. Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

Problem Solution. Let $a = p(\lambda)$ be an eigenvalue of $p(T)$. Thus there exists some vector v such that

$$p(T)v = av = p(\lambda)v.$$

This indicates that

$$\begin{aligned} (a_0 + a_1 T + a_2 T^2 + \cdots + a_m T^m)v &= (a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_m \lambda^m)v \\ a_0 v + a_1 T v + a_2 T^2 v + \cdots + a_m T^m v &= a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \cdots + a_m \lambda^m v \end{aligned}$$

[1] Originally I didn't realize that Axler states (in the beginning of the chapter) that V is assumed to be finite-dimensional. So we lucked out

that is

$$Tv = \lambda v,$$

and λ is an eigenvalue of T.

Conversely if we suppose that λ is an eigenvalue of T, then there is a vector v such that

$$Tv = \lambda v.$$

Thus applying $p(T)$ to v we get the following.

$$\begin{aligned} p(T)v &= (a_0 + a_1 T + a_2 T^2 + \cdots + a_m T^m)v \\ &= a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_m T^m v \\ &= a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \cdots + a_m \lambda^m v \\ &= p(\lambda)v \end{aligned}$$

Hence, $a = p(\lambda)$ is an eigenvalue of $p(T)$.

10 Axler: Chapter 5, Exercise 16

11 Axler: Chapter 5, Exercise 21

Problem Statement. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \ker P \oplus P(V)$.

Problem Solution. [2] Let us refer to $\ker P \oplus P(V)$ as W. Assume by way of contradiction that V is not W. We know that W is a subspace of V since both the image and the kernel of P are subspaces of V, which implies, by our assumption, that V is not a subspace of W. Thus there exists at least one basis vector of V that is not in W. Let v' be such a basis vector, and allow $v_{k1}, \ldots, v_{k2}$ and $v_{i1}, \ldots, v_{i2}$ to be the basis vectors of the kernal and the image of P, respectively.

Since v' is outside of the kernel of P, then Pv' is representable as a linear combination of $v_{i1}, \ldots, v_{i2}$ and since $P^2 = P$, then

$$\begin{aligned} Pv' &= a_{i1}Pv_{i1} + \cdots + a_{i2}Pv_{i2} \\ Pv' - a_{i1}Pv_{i1} - \cdots - a_{i2}Pv_{i2} &= 0 \\ P\left(v' - a_{i1}v_{i1} - \cdots - a_{i2}v_{i2}\right) &= 0. \end{aligned}$$

Hence, $v' - a_{i1}v_{i1} - \cdots - a_{i2}v_{i2}$ is in the kernel of P. Thus we in turn have the following,

$$\begin{aligned} v' - a_{i1}v_{i1} - \cdots - a_{i2}v_{i2} &= a_{k1}v_{k1} + \cdots + a_{k2}v_{k2} \\ v' - a_{i1}v_{i1} - \cdots - a_{i2}v_{i2} - a_{k1}v_{k1} - \cdots - a_{k2}v_{k2} = 0 \end{aligned}$$

however now we have that there is a linear combination of some basis vectors that equal zero, which contradicts the basis vectors' (or any subset) linear independence. Hence our assumption was incorrect and V is indeed W.

12 Finding eigenvalues and eigenvectors.

We are looking for the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

[2] I apologize ahead of time to the reader of this proof. Although I believe it to be correct, I feel like there is a more beautiful proof and that this one can break mirrors with its gaze.

which, seeing as we "don't know" about determinents yet, we will attack by brute force. Solve the equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

for λ and hope we get something good in return. Alas, we get the following two equations

$$\begin{aligned} a + b &= \lambda a \\ a &= \lambda b \end{aligned}$$

from which we can derive that $\lambda^2 = \lambda + 1$, i.e. its the golden ratio!! Thus the two eigenvalues are the roots, φ_+ and φ_-, of the previous equation. Hence any vectors $(a\ b)^T$ that satisfy the above system of equations will be eigenvectors. Furthermore, since the eigenvalues are distinct, then the corresponding eigenvectors will be linearly independent, which in turn implies that the eigenvectors will also be a basis since the vector space in question has dimension two. Hence a matrix of the form

$$\begin{pmatrix} a & a' \\ b & b' \end{pmatrix}$$

where $(a\ b)^T$ and $(a'\ b')^T$ are eigenvectors corresponding to φ_+ and φ_-, respectively, will be invertible according to Artin [1, pg 96]. Continuing this run of "furthermores" and "hences", we thus have that A has a diagonal matrix with respect to the basis of its eigenvectors, as per Axler [2, pg 88-89].

Let's let b of the eigenvectors $(a\ b)^T$ be 1 for both eigenvectors corresponding to φ_+ and φ_-. Hence our eigenvectors are

$$\begin{pmatrix} \varphi_+ \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varphi_- \\ 1 \end{pmatrix}$$

Saving you the eye-sore of a calulation that is finding the inverse of

$$M = \begin{pmatrix} \varphi_+ & \varphi_- \\ 1 & 1 \end{pmatrix} \tag{12.1}$$

via the stick-the-identity-matrix-on-the-right-and-convert-to-rref method, we have that M has

$$M^{-1} = \begin{pmatrix} \frac{1}{\varphi_+ - \varphi_-} & \frac{-1}{\varphi_+/\varphi_- - 1} \\ \frac{-1}{\varphi_+ - \varphi_-} & 1 + \frac{1}{\varphi_+/\varphi_- - 1} \end{pmatrix}$$

as an inverse. Man that's nasty.[3]

Now we have $M^{-1}AM$ will be a diagonal matrix equal to

$$D = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix}.$$

Thus we can write a closed form expression of A^t by MD^tM^{-1}, or in a way more conducive to understanding the computational simplicity,

$$M \begin{pmatrix} \varphi_+^t & 0 \\ 0 & \varphi_-^t \end{pmatrix} M^{-1}$$

See the appendix for the Octave output showing the differences between the one form and the other.

References

[1] Artin, Michael. Algebra. Prentice Hall. Upper Saddle River NJ: 1991.

[2] Axler, Sheldon. Linear Algebra Done Right 2nd Ed. Springer. New York NY: 1997.

[3] A fore-warning to those finding the inverse of M in 12.1. Frequent use of the various forms of the quadradic pertaining to the golden ratio in order to get a nicer-looking matrix may hinder your attempt to calulate the inverse of M, causing undesirable time-loss due to petty mistakes. I could, of course, just be a simplification-idiot.

Appendix

We see below that all of the answers turn out to be the same whether calculating A^t directly or by using its representation in the basis of it's eigenvectors.

```
octave:1> A = [ 1 1 ; 1 0 ];
octave:2> varphip = (1+sqrt(5))/2;
octave:3> varphim = (1-sqrt(5))/2;
octave:4> D = [ varphip 0 ; 0 varphim ];
octave:5> M = [ varphip varphim ; 1 1 ];
octave:6> Minv = M^-1;
octave:7> [ A (M*D*Minv) ]
ans =

   1.00000   1.00000   1.00000   1.00000
   1.00000   0.00000   1.00000  -0.00000

octave:8> [ A^2 (M*D^2*Minv) ]
ans =

   2.00000   1.00000   2.00000   1.00000
   1.00000   1.00000   1.00000   1.00000

octave:9> [ A^3 (M*D^3*Minv) ]
ans =

   3.0000   2.0000   3.0000   2.0000
   2.0000   1.0000   2.0000   1.0000

octave:10> [ A^4 (M*D^4*Minv) ]
ans =

   5.0000   3.0000   5.0000   3.0000
   3.0000   2.0000   3.0000   2.0000

octave:11> [ A^5 (M*D^5*Minv) ]
ans =

   8.0000   5.0000   8.0000   5.0000
   5.0000   3.0000   5.0000   3.0000
```