Math 500: Topology
Homework 2

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Problems

P-1

No.

Let $\mathscr{B}$ be a basis for two topological spaces X and Y. Assume that U is open in X. Then for all $x \in U$ there exists a $B \in \mathscr{B}$ such that $x \in B \subset U$, but as $\mathscr{B}$ is a basis for Y too, then U is open in Y as well. Thus $\mathcal{T}_X \subset \mathcal{T}_Y$. By symmetry of this argument we also have $\mathcal{T}_Y \subset \mathcal{T}_X$. Thus the topologies of X and Y are identical.

As topological spaces, X and Y could still be different even if they have the same topology; X and Y as sets can be distinct. However, this possibility is quickly squandered as we see that any $x \in \bigcup_{B\in\mathscr{B}} B$ will be in both X and Y since each B is a subset of both X and Y and any $x \in X$ will be in $\bigcup_{B\in\mathscr{B}} B$ as per the definition of a basis. This informs us that the sets X and Y must be equal.

Finally, given that both the sets and the topologies of two topological spaces with the same basis must be equal, then the topological spaces must be the same.

P-2

Define a collection of subsets of the integers, $\mathcal{T}$, by the following.

$$\mathcal{T} = \emptyset \cup \left\{ U \mid U \subset \mathbb{Z} \text{ and } \forall a \in U,\ \exists b > 0 \text{ such that } N_{a,b} \subset U \right\}$$

where $N_{a,b}$, as described in the problem, is $\{a + nb \mid n \in \mathbb{Z}\}$ for some fixed $a, b \in \mathbb{Z}$. Prior to proving the infinitude of the prime numbers, we can prove some lemmas which will help us in the proof, and more importantly dispell confusion by diminishing clutter.

Lemma P-2.1 *The collection of subsets of integers, $\mathcal{T}$, defined above is a topology on $\mathbb{Z}$.*

Proof. Certainly the empty set is in $\mathcal{T}$. Also, $\mathbb{Z}$ is contained in the collection as any $N_{a,b}$ is a subset of $\mathbb{Z}$. The union of any collection of open sets $\{U_\alpha\}$ will be open since the sets $N_{a,b}$ which satisfy the requirement for openness of each a in each U_α will also satisfy the requirement for each a in $\bigcup_\alpha U_\alpha$. The intersection of any finite collection $\{U_i\}$ will have that an $a \in \bigcap_i U_i$ will be in all U_i, so the requirement for the existance of an $N_{a,b}$ will be satisfied by all U_i and therefore for $\bigcap_i U_i$. $\square$

Lemma P-2.2 *Unless it is the empty set, every element of $\mathcal{T}$ is infinite.*

Proof. Let U be a nonempty element of $\mathcal{T}$. Then for an $a \in U$, there is an $N_{a,b}$ which is a subset of U for a positive b. Since b is nonzero then $N_{a,b}$ will be infinite, and hence U will be too. $\square$

Lemma P-2.3 *All the sets $N_{a,b}$ are closed in $(\mathbb{Z}, \mathcal{T})$.*

Proof. We can take note that the sets $N_{a,b}$ are the equivalency classes of $a \mod b$. Since these partition the integers, then given a $N_{a,b}$, any $a' \in \mathbb{Z} \setminus N_{a,b}$ will have $a' \not\equiv a \mod b$, implying that $N_{a',b} \subset \mathbb{Z} \setminus N_{a,b}$. This yields to us the openess of $\mathbb{Z} \setminus N_{a,b}$. Hence $N_{a,b}$ is closed. $\square$

Finally, for the epic conclusion, assume for later contradiction that the set of prime numbers, $\mathbb{P}$, is finite. If we notice that the set $N_{a,a} = \{an + a = a(n+1) \mid n \in \mathbb{Z}\}$ is simply the set of all multiples of a, i.e. $a\mathbb{Z}$, then for $U = \{-1, 1\}$, $\mathbb{Z} \setminus U = \{\ldots, -4, -3, -2, 0, 2, 3, 4, \ldots\}$ will be covered entirely by

$$2\mathbb{Z} \cup 3\mathbb{Z} \cup \ldots \cup \hat{p}\mathbb{Z} = N_{2,2} \cup N_{3,3} \cup \ldots \cup N_{\hat{p},\hat{p}}$$

where $\hat{p}$ is set to the largest prime number (the primes were assumed finite). This will cover because all integers larger than $\hat{p}$ will be divisible by an integer less than or equal to $\hat{p}$. Lemma P-2.1 and Lemma P-2.3 thus give us that $\bigcup_{n\in 2,3,\ldots,\hat{p}} N_{n,n}$ is closed since it is a finite union of closed sets. This then implies that U, its complement, is open, but the finitute and non-emptiness of U contradicts Lemma P-2.2. Hence the primes are infinite.

P-3 Munkres §13 exercise 8(b)

Let $\mathbb{R}_\ell$ be the lower-limit topology on $\mathbb{R}$ and $\mathbb{Q}_\ell$ be the lower-limit topology on $\mathbb{Q}$, that is the following collection of sets are a basis for $\mathbb{Q}_\ell$.

$$\big\{[a,b)|a<b \text{ and } a,b\in\mathbb{Q}\big\}$$

Since $\mathbb{R}\supset\mathbb{Q}$, then due to the similarities in the definitions, this implies $\mathbb{R}_\ell\supset\mathbb{Q}_\ell$. Thus to show the topologies different, it suffices to show that $\mathbb{Q}_\ell$ is strictly courser than $\mathbb{R}_\ell$. Let $\mathscr{B}$ and $\mathscr{C}$ be the bases for $\mathbb{R}_\ell$ and $\mathbb{Q}_\ell$, respectively. Let $[a,b)\in\mathscr{B}$ such that a is irrational. If $\mathbb{Q}_\ell$ were at least as fine as $\mathbb{R}_\ell$, then there would be a $[a',b')\in\mathscr{C}$ such that $a\in[a',b')\subset[a,b)$ as per Lemma 13.3 of Munkres. However, as a is irrational and a' is rational, then $a\in[a',b')$ would imply $a'<a$, in which case $[a',b')$ would not be a subset of $[a,b)$. So Lemma 13.3 gives use that $\mathbb{Q}_\ell$ is strictly courser than $\mathbb{R}_\ell$.

P-4 Munkres §16 exercise 1

Let Y be a subspace of X and A a subset of Y. Let $\mathscr{B}_X$ be a basis for X. This yields $\mathscr{B}_Y=\{B\cap Y|B\in\mathscr{B}_X\}$ as the basis inherited by Y. Then the basis inherited from Y by A, call it $\mathscr{B}_A$, is $\{B\cap A|B\in\mathscr{B}_Y\}$. But given the previously mentioned form of each element of $\mathscr{B}_Y$, then $\mathscr{B}_A=\{(B\cap Y)\cap A|B\in\mathscr{B}_X\}$. Thus by the associativity of $\cap$ and since $A\subset Y$ we have $\mathscr{B}_A=\{B\cap A|B\in\mathscr{B}_X\}$. Therefore the topologies inherited by A as a subset of Y or X are identical.

P-5

We will evaluate $\mathbb{R}\times\{0\}$ as a subset of $\mathbb{R}^2$; note that we could have arbitrarily chosen $\{0\}\times\mathbb{R}$. Thus if $\mathscr{B}_S$ is a basis for the standard topology on $\mathbb{R}$ then the topology on $\mathbb{R}\times\{0\}$ as a subspace of $\mathbb{R}^2$ is

$$\big\{(B\times B')\cap(\mathbb{R}\times\{0\})|B,B'\in\mathscr{B}_S\big\}$$

but this is simply

$$\big\{(B\cap\mathbb{R})\times(B'\cap\{0\})|B,B'\in\mathscr{B}_S\big\}=\big\{B\times(B'\cap\{0\})|B,B'\in\mathscr{B}_S\big\}$$

We should be concerned about the possibility of $B'\times\{0\}$ being empty, but this is really not a problem as $B\times\emptyset=\emptyset$ and, in light of Munkres' Lemma 13.1, the addition of the empty set to a basis leaves the generated topology unaltered. With this in mind, the basis for $\mathbb{R}\times\{0\}$ boils down to

$$\{B\times\{0\}|B\in\mathscr{B}_S\}$$

which is certainly isomorphic to $\mathscr{B}_S$, and so the two resulting topologies "coincide".

P-6 Munkres §16 exercise 4

Let X and Y be topological spaces with bases $\mathscr{B}_X$ and $\mathscr{B}_Y$, respectively, and $X\times Y$ is the induced product topology, the basis for which we will denote $\mathscr{B}_X\times\mathscr{B}_Y$. Let the maps π_1 and π_2 be the projection maps of $X\times Y$ as defined in Munkres. Let U be an open set $X\times Y$ and $x\in\pi_1(U)$. Then for all $y\in Y$, $(x,y)\in U$. Thus for any given y there exists a $B_X\times B_Y\in\mathscr{B}_X\times\mathscr{B}_Y$ such that $(x,y)\in B_X\times B_Y\subset U$. This implies $x\in B_x\subset\pi_1(U)$, i.e. $\pi_1(U)$ is open in X. Thus π_1 is an open map.

The same argument holds to show π_2 is an open map, just working with the second position of elements of $X\times Y$ rather than the first.

P-7 Munkres §17 exercise 6

Let the sets A, B, and A_α be subsets of some topological space, X.

(a) $A \subset B \implies \overline{A} \subset \overline{B}$

Assume that $A \subset B$. Let x be in the closure of A, i.e. it's in A, one of its limit points, or both. If x were to be in A, then it is also in B since $A \subset B$, and thus x is also in the closure of B. If x were to instead be in A', then for each neighborhood N, with $x \in N$, there exists some $a \in A \cap N \setminus \{x\}$, but since $A \subset B$, $a \in B \cap N \setminus \{x\}$ as well. This yields that x is also a limit point of B. Thus x is in the closure of B. Hence whether x is in A, just one of its limit points, or both, x will also be in the closure of B. Thus $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

By definition, $\overline{A \cup B}$ is $(A \cup B) \cup (A \cup B)'$ and $\overline{A} \cup \overline{B}$ is $(A \cup A') \cup (B \cup B') = (A \cup B) \cup (A' \cup B')$, so one can prove that the closure operation distributes over a union of two sets by proving only that the set of limit points of two sets is the union of the limit points of each set. Proceeding thusly, we have the following.

Let $x \in (A \cup B)'$. Then for every neighborhood N of x there exists a $y \in A \cup B$ such that $y \in N \setminus \{x\}$. If y is in A, then $x \in A' \subset A' \cup B'$. On the other hand, if y is in B, then $x \in B' \subset A' \cup B'$. Thus whether y is in A, B, or both, x will be in the union of their limit point set, $A' \cup B'$; so $(A \cup B) \subset A' \cup B'$.

Now let $x \in A' \cup B'$. If x is a limit point of A, then for all neighborhoods N of x, $A \cap (N \setminus \{x\})$ is nonempty, but since $A \subset A \cup B$ then $(A \cup B) \cap (N \setminus \{x\})$ is also nonempty. Thus x is a limit point of $A \cup B$. As $\cup$ is commutative, the same argument holds in the case of x being a limit point of B. Therefore whether x is in A', B', or both, it is also a limit point of $A \cup B$; hence $A' \cup B' \subset (A \cup B)'$.

Given that $(A \cup B) \subset A' \cup B'$ and $A' \cup B' \subset (A \cup B)'$, then $(A \cup B) = A' \cup B'$, which gives us our ultimate result of $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(c) $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$

Let $x \in \cup\overline{A_\alpha}$. Then x is a limit point of some $A_{\alpha'}$. Thus every neighborhood N of x, $A_{\alpha'} \cap (N \setminus \{x\})$ is nonempty, but since $A_{\alpha'} \subset \cup A_\alpha$ then $(\cup A_\alpha) \cap (N \setminus \{x\})$ is also nonempty. Therefore $x \in \overline{\cup A_\alpha}$, which implies $\cup\overline{A_\alpha} \subset \overline{\cup A_\alpha}$

Equivalency Counter-example. In $\mathbb{R}$ with the standard topology, let $\{U_\alpha\}$ be the set

$$\left\{ ({}^1\!/n, 2) \,\middle|\, n \in \mathbb{Z}^+ \right\}.$$

Given this, $\bigcup U_\alpha = (0, 2)$ and its closure is $[0, 2]$, however $\left\{\overline{U_\alpha}\right\}$ is

$$\left\{ [{}^1\!/n, 2] \,\middle|\, n \in \mathbb{Z}^+ \right\}$$

which contains no interval that contains 0. Hence $0 \notin \bigcup \overline{U_\alpha}$, and therefore the inclusion given by $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$ is strict.

EXTRA

Extra-1 Describe all topological structures having exactly one basis.

To make our discussion clear, we'll use $\mathcal{T}$ to refer the topology of any such topological structures.

We'll first note that since $\mathcal{T}$ of a topological space is always a basis for the space. With this we can quickly narrow our search for topological structures having only one basis to structures for which any strict subcollection of $\mathcal{T}$ cannot be a basis for the topological space. Given this realization and Munkres' Theorem 13.1, a topological space with only one basis must have that $\mathcal{T}$ has no subcollection which can generate it under unions. Another way of saying this is that the union of any number of sets in $\mathcal{T}$, $\bigcup_\alpha U_\alpha$, must be contained within the collection of the sets in the union, i.e. $\bigcup_\alpha U_\alpha \in \{U_\alpha\}$ must hold true. Otherwise we under the auspice of Munkres' Theorem 13.1, we could remove the resulting union to obtain the basis $\mathcal{T} \setminus \left\{\bigcup_\alpha U_\alpha\right\}$. This is a requirement of all topological spaces having only one basis. We prove the sufficiency of this property next, thereby revealing that this property is a necessary and sufficient condition of all topological structures with exactly one basis.

Let X be a topological space with topology $\mathcal{T}$ such that

$$\forall \{U_\alpha\} \subset \mathcal{T},\ \bigcup_\alpha U_\alpha \in \{U_\alpha\} \tag{Extra-1.1}$$

Assume by way of contradiction that $\mathscr{B}_1$ and $\mathscr{B}_2$ are distinct bases of X. Without loss of generality let $B \in \mathscr{B}_1$ but $B \notin \mathscr{B}_2$. Since basis elements are also open sets, then Munkres' Theorem 13.1 gives us that there exists a collection $\{B_\alpha\} \subset \mathscr{B}_2$ such that $B = \bigcup_\alpha B_\alpha$. Thus the property in Extra-1.1 assumed of X tells us that $B \in \mathscr{B}_2$, which contradicts the definition of B. Hence $\mathscr{B}_1 = \mathscr{B}_2$.