Math 500: Topology
Homework 4

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Problems

P-1 Munkres §19 exercise 6

Let $\{\mathbf{x}_i\}$ be a sequence in $\prod X_\alpha$ and $\mathbf{x}$ and element of $\prod X_\alpha$.

Assume that $\{\mathbf{x}_i\}$ converges to $\mathbf{x}$. Let N be a neighborhood of $\pi_\beta(\mathbf{x})$ for some β in the index set of $\prod X_\alpha$. Then $U \subset \prod X_\alpha$ where $\pi_\beta(U) = N$ and $\pi_\alpha(U) = X_\alpha$ for $\alpha \neq \beta$ is a neighborhood of $\mathbf{x}$ since it is open in the product topology and contains $\mathbf{x}$. Hence there are infinitely many $\mathbf{x}_j$ of $\{\mathbf{x}_i\}$ which are also in U, however each $\mathbf{x}_j$ has that $\pi_\beta(\mathbf{x}_j) \in \pi_\beta(U) = N$. Thus N contains infinitely many points of $\{\pi_\beta(\mathbf{x}_i)\}_i$, so $\pi_\beta(\mathbf{x}_i) \to \pi_\beta(\mathbf{x})$.

Conversely, assume that for all β in the index set of $\prod X_\alpha$ the sequence $\{\pi_\beta(\mathbf{x}_i)\}$ converges to $\pi_\beta(\mathbf{x})$. Let V be a neighborhood of $\mathbf{x}$. Then there is a basis element B of $\prod X_\alpha$ with $x \in B \subset V$. Being in the product topology, only finitely many coordinates of B are sets which are not X_α. Letting $\pi_{\alpha_1}(B), \ldots, \pi_{\alpha_m}(B)$ be these sets, we have, by assumption, that there exist $N_1, \ldots, N_m$ such that all $n \geq N_j$ has that $\pi_{\alpha_j}(\mathbf{x}_n) \in \pi_{\alpha_j}(B)$. Thus if we set

$$N = \max\{N_j\} \tag{P-1.1}$$

then $\pi_\alpha(\mathbf{x}_n) \in \pi_\alpha(B)$ for all α in $\{\alpha_1, \ldots, \alpha_m\}$ and $n \geq N$. From this we get that $\mathbf{x}_n \in B \subset V$ for all $n \geq N$, which implies the convergence of $\{\mathbf{x}_i\}$ to $\mathbf{x}$.

Does the same hold for the box topology? The latter half of the above proof is dependent on the finititude of the number of coordinate sets in a given basis element of the product topology on $\prod X_\alpha$. As exemplified by the following scenario, the box topology may disallow such an N as in P-1.1 above.

Let $\prod X_\alpha$ be $\mathbb{R}^\omega$ with the box topology and define the sequence $\{\mathbf{x}_i\}$ by $\mathbf{x}_i = ({}^1\!/\!_i, {}^2\!/\!_i, {}^3\!/\!_i \ldots)$. So for any coordinate j, $(\pi_j(\mathbf{x}_1), \pi_j(\mathbf{x}_2), \pi_j(\mathbf{x}_3), \ldots) = ({}^j\!/\!_1, {}^j\!/\!_2, {}^j\!/\!_3, \ldots)$ and certainly the right hand side converges to 0. Thus $\mathbf{x} = (0, 0, 0, \ldots)$ will have the property that for each j, $\pi_j(\mathbf{x}_i) \to \pi_j(\mathbf{x})$. However, for the neighborhood $B = (-1, 1) \times (-1, 1) \times (-1, 1) \times \cdots$ of $\mathbf{x} = (0, 0, 0, \ldots)$ there exists no N such that all $n \geq N$ has $\mathbf{x}_n \in B$ since $\pi_N(\mathbf{x}_N) = {}^N\!/\!_N = 1 \notin (-1, 1)$ and thus $\mathbf{x}_N \notin B$.

P-2

The proof of the first part of Munkres' Theorem 20.4 shows that $\mathcal{T}_{\text{prod}} \subset \mathcal{T}_{\text{uniform}} \subset \mathcal{T}_{\text{box}}$. So it only remains to be shown that the box topology has a basis element for which no basis element of the uniform topology is contained within, and that the uniform topology contains a basis element for which no basis element of the product topology is contained within.

In the case of $\mathbb{R}^J$ where J is infinite, set B to be a basis element of the box topology defined as follows: for each $n \in \mathbb{Z}^+$ there exists a unique $\alpha \in J$ such that $\pi_\alpha(B) = (-{}^1\!/\!_n, {}^1\!/\!_n)$ and if α' is not such an α, then $\pi_{\alpha'}(B) = \mathbb{R}$. Note that for the following example the $\alpha \in J$ for which $\pi_\alpha(B) = \mathbb{R}$ and for which $\pi_\alpha(B) \neq \mathbb{R}$ do not matter, just that there are a countably infinite number of them. So for $\mathbf{0} \in B$, the sequence of all zeros, there is no open ball $B_{\overline{\rho}}(\mathbf{0}, r)$ of the uniform topology such that $\mathbf{0} \in B_{\overline{\rho}}(\mathbf{0}, r) \in B$ since for any n such that ${}^1\!/\!_n < r$, an $\alpha \in J$ can be found such that $B_{\overline{d}}(0, r) \not\subset \pi_\alpha(B) = (-{}^1\!/\!_n, {}^1\!/\!_n)$, since, say, $\frac{r + 1/n}{2}$ is in $B_{\overline{d}}(0, r)$ but not in $(-{}^1\!/\!_n, {}^1\!/\!_n)$. Thus the uniform and box topologies are different.

To see that the uniform topology on $\mathbb{R}^J$ is different from the product topology, we need only look at an open ball of the uniform topology with radius $r < 1$. This will be the set $\prod_\alpha B_{\overline{d}}(x_\alpha, r)$ for some center $(\mathbf{x}_\alpha)$, but no basis element of the product topology can be contained within this set since no basis element B of the product topology can have more than a finitie number of $\alpha \in J$ with $\pi_\alpha(B) \neq \mathbb{R}$.

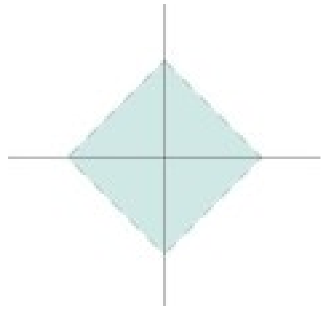

Figure 1: An arbitrary ball of the metric d'.

P-3 Munkres §20 exercise 1(a)

Let $d' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

With this definition we see that d' is a metric by

(1) $d'(\mathbf{x}, \mathbf{x}) = 0$ since $x_i - x_i = 0$

(2) $d'(\mathbf{x}, \mathbf{y}) > 0$ for all $x \neq y$ since d' is the sum of absolute values

(3) $d'(\mathbf{x}, \mathbf{y}) = d'(\mathbf{y}, \mathbf{x})$ since $|x_i - y_i| = |y_i - x_i|$ for each i

(4) $d'(\mathbf{x}, \mathbf{z}) \leq d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z})$ since $d'(\mathbf{x}, \mathbf{y}) + d'(\mathbf{y}, \mathbf{z}) = d(x_1, y_1) + \cdots + d(x_n, y_n) + d(y_1, z_1) + \cdots + d(y_n, z_n) = \big(d(x_1, y_1) + d(y_1, z_1)\big) + \cdots + \big(d(x_n, y_n) + d(y_n, z_n)\big) \geq d(x_1, z_1) + \cdots + d(x_n, z_n) = d'(\mathbf{x}, \mathbf{z})$, where d is the euclidean metric on $\mathbb{R}$.

Now that we have that d' is indeed a metric we can compare its induced topology with that of the square metric, ρ, on $\mathbb{R}^n$ via Munkres' Lemma 20.2. Clearly for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\rho(\mathbf{x}, \mathbf{y}) = \max_i\{|x_i - y_i|\} \leq \sum_i |x_i - y_i| = d'(\mathbf{x}, \mathbf{y}) \tag{P-3.2}$$

and

$$d'(\mathbf{x}, \mathbf{y}) = \sum_i |x_i - y_i| \leq n \max_i\{|x_i - y_i|\} = n\rho(\mathbf{x}, \mathbf{y}) \tag{P-3.3}$$

By Equation P-3.2, inside of any ϵ-ball $B_\rho(\mathbf{x}, \epsilon)$ is one $B_{d'}(\mathbf{x}, \epsilon)$. So Munkres' Lemma 20.2 tells us that the topology induced by d' is finer than the one induced by ρ.

By Equation P-3.3, any $B_{d'}(\mathbf{x}, \epsilon)$ has a $B_\rho(\mathbf{x}, \epsilon/n)$ contained within it. Again turning to Munkres' Lemma 20.2, we get that the topology induced by ρ is finer than that of d'.

Given the above two results, d' and ρ induce the same topology on $\mathbb{R}^n$, that is, the usual topology.

Sketching. Figure 1 shows a ball for the metric d' in $\mathbb{R}^2$. It is no particular ball.

P-4 Munkres §23 exercise 2

Let $\{A_n\}$ be a collection of connected subspaces of X such that $A_n \cap A_{n+1} \neq \emptyset$. For our base case, A_1 is connected by assumption. Assume that for 1 through $n-1$ we have $\bigcup_{i=1,\ldots,n-1} A_i$ is connected. Assume for later contradiction that C, D separate $\bigcup A_n$. By Munkres Lemma 23.2, $\bigcup_{i=1,\ldots,n-1} A_i$ is contained within C or D, so without loss of generality allow $\bigcup_{i=1,\ldots,n-1} A_i \subset C$. By the same Lemma, A_n must also be contained within one of C or D, however, A_n has nonempty intersection A_{n-1} and therefore with $\bigcup_{i=1,\ldots,n-1} A_i$ as well. Thus since $\bigcup_{i=1,\ldots,n-1} A_i$ is contained in C, A_n must also be, thereby yielding that D actually is the emptyset. This contradicts C and D separating $\bigcup A_n$ and therefore $\bigcup A_n$ must be connected.

Extra

EX-1

Our motivation for this is to envision that there is a circular, impenetrable barrier around the origin of $\mathbb{R}^2$. Normally, one would represent "impenetrable" mathematically through the use of ∞, in our case we would set the distance between certain points to ∞ but since the range of a metric must be $\mathbb{R}$ this cannot be done. So we turn to the standard bounded metric $\overline{d}$ corresponding to the euclidean metric on $\mathbb{R}^2$ for aide as it facilates a pseudo-infinity.

Let C be the disc centered at the origin, $\{(x, y) \in \mathbb{R}^2 \ : \ |x| + |y| \leq {}^1\!/\!_2\}$. Define d (our metric in question) on $\mathbb{R}^2$ by the following

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} \overline{d}(\mathbf{x}, \mathbf{y}) & \mathbf{x} \in C \text{ and } \mathbf{y} \in C \\ \overline{d}(\mathbf{x}, \mathbf{y}) & \mathbf{x} \notin C \text{ and } \mathbf{y} \notin C \\ 2 & \text{otherwise} \end{cases}$$

This 2 acts as our infinity, since the standard bounded metric has a maximum value of 1; regarding our original motivation, this metric makes the border of C "impenetrable" resulting in "infinite" distance of 2.

Metric Proof. Despite calling it a metric already, we have yet to prove that d is indeed one. Alas it is:

(1) $d(\mathbf{x}, \mathbf{x}) = 0$ since $\mathbf{x} \in C$ or not, but in both cases $d(\mathbf{x}, \mathbf{x}) = \overline{d}(\mathbf{x}, \mathbf{x}) = 0$

(2) $d(\mathbf{x}, \mathbf{y}) > 0$ for all $x \neq y$ since d is $\overline{d}$ if $\mathbf{x}$ and $\mathbf{y}$ are both inside of or both outside of C, else $d(\mathbf{x}, \mathbf{y}) = 2$

(3) $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ since d is $\overline{d}$ if $\mathbf{x}$ and $\mathbf{y}$ are both inside of or both outside of C, else $d(\mathbf{x}, \mathbf{y}) = 2$ and $d(\mathbf{y}, \mathbf{x}) = 2$

(4) d obeys the triangle inequality for $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as the following "truth" table demonstrates, where for $\mathbf{x}, \mathbf{y}, \mathbf{z}$ 0 means outside of C and 1 means inside of C.

$\mathbf{x}$	$\mathbf{y}$	$\mathbf{z}$	$d(\mathbf{x}, \mathbf{y})$	$d(\mathbf{y}, \mathbf{z})$	$d(\mathbf{x}, \mathbf{z})$	Does triangle inequality hold?
0	0	0	$\overline{d}(\mathbf{x}, \mathbf{y})$	$\overline{d}(\mathbf{y}, \mathbf{z})$	$\overline{d}(\mathbf{x}, \mathbf{z})$	yes
0	0	1	$\overline{d}(\mathbf{x}, \mathbf{y})$	2	2	yes
0	1	0	2	2	$\overline{d}(\mathbf{x}, \mathbf{z})$	yes
0	1	1	2	$\overline{d}(\mathbf{y}, \mathbf{z})$	2	yes
1	0	0	2	$\overline{d}(\mathbf{y}, \mathbf{z})$	2	yes
1	0	1	2	2	$\overline{d}(\mathbf{x}, \mathbf{z})$	yes
1	1	0	$\overline{d}(\mathbf{x}, \mathbf{y})$	2	2	yes
1	1	1	$\overline{d}(\mathbf{x}, \mathbf{y})$	$\overline{d}(\mathbf{y}, \mathbf{z})$	$\overline{d}(\mathbf{x}, \mathbf{z})$	yes

The Finale. Now with this metric d we can set $B_1 = B_d\big((0,0), {}^3\!/\!_4\big)$ and $B_2 = B_d\big(({}^1\!/\!_2, 0), {}^7\!/\!_8\big)$. So both $(0,0)$ and $({}^1\!/\!_2, 0)$ are in C, and clearly $B_1 = C$ and $B_2 \subset C$ since they cannot contain any points outside of C due to their radii being less than 2. However despite the fact that B_2 has a larger radius of ${}^7\!/\!_8$, it doesn't contain the point $(-{}^1\!/\!_2, 0)$ of C since this is of distance 1 away from the center of B_2, $(0, {}^1\!/\!_2)$. Hence $B_2 \subsetneq B_1$.