Math 500: Topology
Homework 6

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January 12, 2013
http://coursework.tylerlogic.com/courses/math500/homework06

Problems

1

The quotient space, X^*, induced by the equivalence relation on $X = \mathbb{R}^2$

$$(u, v) \sim (x, y) \text{ if } u^2 + v^2 = x^2 + y^2$$

is homeomorphic to $\mathbb{R}_+$. At a bird's-eye view, the equivalence classes will be the radii of concentric circles centered at the origin of $\mathbb{R}^2$. To *convince* ourselves of this homeomorphism, we will make use of on Munkres' Corollary 22.3 by defing a satisfying g and proving that it is a quotient map.

Defining a g Let $g : \mathbb{R}^2 \to \mathbb{R}_+$ be defined by $(x, y) \mapsto x^2 + y^2$, that is, g maps a point to the square of its distance from the origin.

g is a quotient map The map g is surjective as any $x \in \mathbb{R}_+$ is mapped to by $(0, \sqrt{x})$. It is also continuous since it's the composition of two continuous functions: the addition function and the component-wise squaring function, both on $\mathbb{R}^2$. In other words g is $(\cdot + \cdot) \circ ((\cdot)^2, (\cdot)^2)$ with the adddition function being continuous by Munkres' Lemma 21.4 and the component-wise squaring function being continuous by Munkres' Theorem 18.4. We these two properties, we are left only to prove $U \in \mathbb{R}_+$ is open whenever $g^{-1}(U)$ is open in order to know that g is a quotient map.

So assume that $g^{-1}(U)$ is an open set of $\mathbb{R}^2$. Let r^2 be an element of U for some $r \geq 0$. Then as g is surjective, there is a $(x, y) \in g^{-1}(U)$ which maps to r^2, i.e. $x^2 + y^2 = r^2$. Since $g^{-1}(U)$ is open then there is a ball B, say with radius d, centered at (x, y) which is contained in $g^{-1}(U)$. Therefore $B' = (r^2 - d, r^2 + d)$ is an open interval containing r^2. Thus any element z^2 of B' with $z \geq 0$ has

$$|r^2 - z^2| < d \tag{1.1}$$

Letting θ be the angle at which (x, y) is from the x-axis, we see that the point $(z \cos\theta, z \sin\theta)$ is contained within B by the following

$$\begin{aligned}
\sqrt{(r\cos\theta - z\cos\theta)^2 + (r\sin\theta - z\sin\theta)^2} &= \sqrt{(r - z)^2(\cos^2\theta + \sin^2\theta)} \\
&= \sqrt{(r - z)^2} \\
&= |r - z| \\
&\leq |r^2 - z^2| \\
&< d
\end{aligned}$$

making use of Equation 1.1 for the last deduction. Thus since $g(z\cos\theta, z\sin\theta) = (z\cos\theta)^2 + (z\sin\theta)^2 = z^2(\cos^2\theta + \sin^2\theta) = z^2$, then we conclude that for every point in B' there is a point of B which maps to it, i.e. B' is completely contained in U and thus the openess of U is implied.

With this and the facts that g is surjective and continuous, then g is a quotient map. Finally since $g^{-1}(r)$ is the set of all points of distance $\sqrt{r}$ from the orgin then $X^* = \{g^{-1}(r) | r \in \mathbb{R}_+\}$. Combining these results, we appeal to Corollary 22.3 to give us that g induces a homeomorphism between X^* and $\mathbb{R}_+$.

2

Assume that Y is a closed subspace of a normal space X. Thus closed subsets A and B of Y are also closed in X. By the normalcy of X, we can therefore separate A and B by disjoint open sets of X, say U and V, respectively. So because A and B are contained in Y, then the open sets $U \cap Y$ and $V \cap Y$ of Y separate A and B and are disjoint since $(U \cap Y) \cap (V \cap Y) = Y \cap (U \cap V) = Y \cap \emptyset = \emptyset$. Therefore Y is normal.

3 Prove Urysohn's Lemma for metric space (X, d)

Define the function $d_A : X \to \mathbb{R}_+$, where A is some subset of X, by

$$x \longmapsto \inf\{d(x,a) \mid a \in A\}.$$

This definition yields the following lemma.

Lemma 3.1 *For a subset A of (X, d), the function d_A is continuous.*

Proof. We must consider both forms of basis elements of $\mathbb{R}_+$, (u, v) and $[0, v)$. If $x \in d_A^{-1}(u, v)$, then for $y \in B(x, \delta)$ where $\delta = \min(d_A(x) - u, v - d_A(x))$ we have

$$d_A(y) \geq d_A(x) - d(x, y) > d_A(x) - \delta \geq u$$

and

$$d_A(y) \leq d_A(x) + d(x, y) < d_A(x) + \delta \leq v$$

which implies that $d_A(y)$ is also in (u, v). Hence the entirety of $B(x, \delta)$ is contained within $d_A^{-1}(u, v)$. On the other if $x \in d_A^{-1}[0, v)$, then the previous argument will hold for $d_A(x) \neq 0$. So when $d_A(x) = 0$ then any element of $B(x, v)$ is within a distance of v from an element of A, namely x, so $B(x, v) \in d_A^{-1}([0, v))$.

Thus the preimage of d_A for any basis element of $\mathbb{R}_+$ will be open since in any case we can find a ball contained within the preimage. Therefore d_A is continuous. □

Now we can define $f_{AB} : X \to [0, 1]$ by

$$x \longmapsto \frac{d_A(x)}{d_A(x) + d_B(x)}$$

for two disjoint closed subsets of X, A and B. The denominator will not be zero since if that were true, an element of X would have to be contained in both A and B or at least be a limit point of them both; but this cannot be the case as the sets are both closed and disjoint. Thus this fact, combined with both Lemma 3.1 and Munkres' Theorem 21.5 informs us that f_{AB} is continuous. Therefore $\overline{f_{AB}} : X \to [a, b]$ defined by $(b-a)f_{AB}+a$ is a continuous function which satisfies the requirements specified in Urysohn's Lemma.