Math 500: Topology
Homework 8

Lawrence Tyler Rush
<me@tylerlogic.com>

January 12, 2013
http://coursework.tylerlogic.com/courses/math500/homework08

Problems

P-1 Munkres §51 exercise 3 b, c, and d

(b) Contractible spaces are path connected.

Assume X to be a contractible space. Let H be the homotopy between the identity function on X and the constant function c. Letting x, y be elements of X, define the map $f_{xy} : [0,1] \to X$ by

$$f_{xy}(t) = \begin{cases} H(x, 2t) & t \in [0, 1/2] \\ H(y, 2(1-t)) & t \in [1/2, 1] \end{cases}$$

This map is well-defined since $H(x, 2(1/2)) = H(x, 1) = c$ and $H(y, 2(1-1/2)) = H(y, 1) = c$ according to homotopic nature of H. Furthermore, due to H's continuity f_{xy} is continuous by the pasting lemma. Thus since the domain, $[0,1]$, of f_{xy} is a closed interval and both $f_{xy}(0) = H(x, 2(0)) = H(x, 0) = \mathrm{id}_X(x) = x$ and $f_{xy}(1) = H(y, 2(1-1)) = H(y, 0) = \mathrm{id}_X(y) = y$ then f is a path between x and y. So X is path connected.

(c) $[X, Y]$ contains one element for all X when Y is contractible

Assume that Y is contractible. Showing that $[X, Y]$ has only one element amounts to showing that any two continuous maps from X to Y are homotopic. We proceed thusly.

Let f and g be continuous functions from X to Y. We create a homotopy between them by using the path connectedness of Y, by way of the previous problem. Define $H' : X \times [0,1] \to Y$ by

$$H'(x, t) = \begin{cases} H(f(x), 2t) & t \in [0, 1/2] \\ H(g(x), 2(1-t)) & t \in [1/2, 1] \end{cases}$$

where H is the homotopy between the identity map on Y and some constant function c on Y. As previously, this function is well defined at $t = 1/2$ since $H'(f(x), 2(1/2)) = H(f(x), 1) = c$ and $H'(g(x), 2(1-1/2)) = H(g(x), 1) = c$. Also, H' is continuous by the pasting lemma. Thus because $H'(x, 0) = H(f(x), 0) = \mathrm{id}_Y(f(x)) = f(x)$ and $H'(x, 1) = H(g(x), 0) = \mathrm{id}_Y(g(x)) = g(x)$ then H' is a homotopy between f and g. Thus $f \simeq g$, implying that all continuous elements of Y^X are homotopic. So $[X, Y]$ has a size of one.

(d) Show $[X, Y]$ has one element for contractible X and path connected Y

Assume that X is a contractible space and Y is path connected. Let f, g be continuous maps from X to Y and H be the homotopy between id_X and some constant map c on X. Since Y is path connected we can find a path h between $f(c)$ and $g(c)$. We can manipulate the domain of h as we see fit, so let's make it $[1/3, 2/3]$. With these things, we construct a homotopy essentially by contracting $f(x)$ to $f(c)$ with H, using h to travel to $g(c)$, and then expanding $g(c)$ to $g(x)$ again using H. We do this by defining H' as

$$H'(x, t) = \begin{cases} f(H(x, 3t)) & t \in [0, 1/3] \\ h(t) & t \in [1/3, 2/3] \\ g(H(x, 3(1-t))) & t \in [2/3, 1] \end{cases}$$

This map is well defined since $f(H(x, 3(1/3))) = f(H(x, 1)) = f(c) = h(1/3)$ and $g(H(x, 3(1-2/3))) = g(H(x, 1)) = g(c) = h(2/3)$. Furthermore H' is continuous by the pasting lemma since its constituent parts are either continuous or compositions of continuous functions. Thus $f \simeq g$ via the homotopy of H'. Since f and g were abitrary, then all continous functions from X to Y are homotopic to each other, and therefore $[X, Y]$ has a single element.