Math 500: Topology
Homework 9

Lawrence Tyler Rush
<me@tylerlogic.com>

January 12, 2013
http://coursework.tylerlogic.com/courses/math500/homework09

Problems

P-1

Let $p : E \to B$ be a covering space with E path connected. Letting $b_0 \in B$, Munkres' Theorem 54.4 then informs us that the lifting correspondence, $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$, derived from p is surjective. So if B were to be simply connected, this would then demand that there be only one element in $p^{-1}(b_0)$. So p is bijective, moreover, it's a homeomorphism.

P-2 A continuous, non-surjective map $f : X \to S^2$ is nulhomotopic.

Assume that $f : X \to S^2$ is a continuous map but not surjective. Then there exists some s_0 in S^2 outside of $f(X)$. Fortunately, $S = S^2 \setminus \{s_0\}$ is contractible to any point since the singly-punctured S^2 sphere is homeomorphic to $\mathbb{R}^2$ which can be contracted to a point $\mathbf{x}$ using "the linear combination homotopy" mapping, $(\mathbf{x}', t) \mapsto t\mathbf{x} + (1-t)\mathbf{x}'$. So choosing $s_1 \in f(X)$ there exists a homotopy H between the identity map on S and the constant map on S taking values to s_1. Thus define $H' : X \times I \to S^2$ by

$$H'(x, t) = (j \circ H)(f(x), t)$$

where j is the inclusion map from S to S^2. This map is continuous since the composition of H and the inclusion map is continuous. The map is therefore a homotopy because $H'(x, 0) = (j \circ H)(f(x), 0) = j(f(x)) = f(x)$ and $H'(x, 1) = (j \circ H)(f(x), 1) = j(s_1) = s_1$. Thus f is nulhomotopic.

P-3

We will accomplish our task by stealing Munkres' three-step idea for the figure eight retraction which he sketches in his Figure 58.2. Throughout, the two points of puncture will be q and p. Also, to make the proof easier, we'll assume that $q = (-1, 0)$ and $p = (1, 0)$, so keep watch for ominous uses of 1 and 2 for the construction of the retraction.

Step 1: Shrink $\mathbb{R}^2$ to encompassing disc. Define $r_1 : X \times I \to A$ (the first retraction) by

$$r_1(x, t) = \begin{cases} x & x \in \overline{B}(\mathbf{0}, 2) \\ t\frac{2}{\|x\|}x + (1-t)x & x \in \mathbb{R}^2 \setminus B(\mathbf{0}, 2) \end{cases}$$

The use of the closed ball in the first line of the definition and the complement of the open ball in the second may look a little suspect, but this little trickery simply allows us to make good use of the pasting lemma. That is, these two partitions of the domain are both closed sets with a union of $\mathbb{R}^2$ and intersection of the ball's boundary; and since $t\frac{2}{\|x\|}x + (1-t)x = x$ for all t and x on the boundary of $B(\mathbf{0}, 2)$, then the pasting lemma yields to us the continuity of r_1 since the identity map and scalar multiplication are continuous.

Step 2: Shrink disc to the discs of figure eight. We define $r_2 : \overline{B}(\mathbf{0}, 2) \times I \to \overline{B}(q, 1) \cup \overline{B}(p, 1)$ as the second retraction by

$$r_2((x, y), t) = \begin{cases} (x, y) & (x, y) \in \overline{B}(q, 1) \cup \overline{B}(p, 1) \\ t\left(x, \sqrt{1-(x-1)^2}\right) + (1-t)(x, y) & y \geq 0 \text{ and } (x, y) \in \overline{B}(\mathbf{0}, 2) \setminus \big(B(q, 1) \cup B(p, 1)\big) \\ t\left(x, -\sqrt{1-(x-1)^2}\right) + (1-t)(x, y) & y \leq 0 \text{ and } (x, y) \in \overline{B}(\mathbf{0}, 2) \setminus \big(B(q, 1) \cup B(p, 1)\big) \end{cases}$$

which is again continuous by the pasting lemma since all three partitionings of the domain are closed and any point in the intersection of any two of the partitionings (i.e. points in the boundaries) are mapped to themselves.

Step 3: Push inards of figure eight discs to the exterior. We'll define our last retraction $r_3 : \overline{B}(p,1) \cup \overline{B}(q,1) \times I \to Bd\big(B(p,1) \cup B(q,1)\big)$ by

$$r_3(x,t) = \begin{cases} t(p + \frac{x-p}{\|x-p\|}) + (1-t)x & x \in \overline{B}(p,1) \\ t(q + \frac{x-q}{\|x-q\|}) + (1-t)x & x \in \overline{B}(q,1) \end{cases}$$

This is once again continuous by the pasting lemma since $\overline{B}(p,1)$ and $\overline{B}(q,1)$ are both closed and the only point of their intersection is $\mathbf{0}$ which is map to itself by both constituent maps of r_3.

Putting it together. We'll simply compose all of these continuous functions to another continuous function $r = r_3 \circ r_2 \circ r_1$. So r is continuous, and is our deformation retraction we're looking for.

P-4

This is closely related to the previous two problems. The main idea behind the construction of the deformation retract is to first shrink $\mathbb{R}^3$ with the axes removed, denote it by $\mathbb{R}^3 \setminus xyz$, to S^2 with six punctures, two for each axis. Since the singly-punctured S^2 is homeomorphic to $\mathbb{R}^2$ as mentioned in problem P-2, then the previous problem informs us that we should be able to make a bouquet with the other five punctures, leading to the 5 circle bouquet.

To deformation retract to the hexa-punctured S^2, well use the following deformation retraction

$$(x,t) \mapsto t\frac{x}{\|x\|} + (1-t)x$$

which works because $\mathbf{0} \notin \mathbb{R}^3 \setminus xyz$. After doing this we will choose a point in the resulting punctured sphere to be the center point of the bouquet and retract á la the previous problem, but this time accounting for five points instead of two. Due to time constraints, I couldn't work out the details of the latter.

P-5

Fundamental Group of Solid Torus The fundamental group of S^1 is isomophic to the integers, and D^2 is simply connected. This leads us via Munkres' Theorem 60.1 to the solid torus having a fundamental group of $\mathbb{Z} \times \{e\} \cong \mathbb{Z}$.

Fundamental Group of $S^1 \times S^2$ Likewise, with the fundamental group of S^1 being $\mathbb{Z}$ and S^2 being simply connnected, Munkres' Theorem 60.1 gets us that the fundamental of $S^1 \times S^2$ is again $\mathbb{Z} \times \{e\} \cong \mathbb{Z}$.

Bonus

B-1

A manifold should be regular because, locally, it is inditinguishable from some euclidean space, and regularity is a local property. This idea of regularity simply being a local property is corroborated by Munkres' Lemma 31.1(a) from which we know that inside any neighborhood of a point we can find another neighborhood with a closure contained in the first. A manifold should reflect this property locally because euclidean spaces do.