Math 501: Differential Geometry
Homework 3

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2 do Carmo pg 65 exercise 1

The set S is the cylinder $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by the mapping

$$(x, y, z) \mapsto x^2 + y^2.$$

With this definition, $f^{-1}(1)$ is exactly S since S is the set of all points of $\mathbb{R}$ which have $x^2 + y^2 = 1$. Since the differential $df_{(x_0,y_0,z_0)}$ is the left multiplication by the matrix $(2x_0, 2y_0, 0)$, then the only point critical point of the differential is $(0, 0, 0)$. Putting the previous two ideas together then gives us that 1 is a regular point of the mapping f since $f^{-1}(1) = S$ doesn't contain the origin of $\mathbb{R}^3$. Hence, because f is smooth, then do Carmo's Proposition 2 of Section 2-2 informs us that S is regular.

3 do Carmo pg 65 exercise 2

Denote the open disc, $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } x^2 + y^2 < 1\}$, by D and the closed disc $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } x^2 + y^2 \leq 1\}$, by $\overline{D}$.

Closed Disc The closed disc $\overline{D}$ is not a regular surface because if it were, then by the definition of a regular surface there would exist a neighborhood $V \subset \mathbb{R}^3$ of $(0, 1) \in \overline{D}$ such that $\overline{D} \cap V$ is homeomorphic to some open set of $\mathbb{R}^2$, say U. However, the point $(0, 1)$ is in $\overline{D} \cap V$ but is not an interior point, which contradicts the fact that U, being open in $\mathbb{R}^2$, contains only interior points.

Open Disc Setting $U \subset \mathbb{R}^2$ to the open set $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ and defining $f : U \to \mathbb{R}$ by $f(u, v) = 0$, then D is simply the graph $\{(u, v, f(u, v)) \mid (u, v) \in U\}$ of $\mathbb{R}^3$. So D is a regular surface by do Carmo's first proposition of chapter two.

4 do Carmo pg 66 exercise 11

Let S be the set $\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$.

Regular Surface Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 - y^2 - z$. With this definition, f is smooth and df_p for $p = (x_0, y_0, z_0)$ is the linear map corresponding to the matrix $(2x_0, -2y_0, -1)$. Since the third parameter of the matrix is -1, then df_p is surjective. Since p was arbitrary, we know the differential to be surjective at all points of $\mathbb{R}^3$, and thus all points of $\mathbb{R}$ are regular values of the map f. Hence by do Carmo's second proposition of section 2-2, $f^{-1}(0)$ is a regular surface, but $f^{-1}(0)$ is all points that satify $0 = x^2 - y^2 - z$, which is exactly the set S. So S is regular.

Parametrizations For the following, note that S can equivalently be given by the equation $x^2 - y^2 - z = 0$.

(a)

The parametrization defined by

$$\mathbf{x}(u, v) = (u + v, u - v, 4uv)$$

covers S due to the following.

$$\begin{aligned}(u+v)^2-(u-v)^2-4uv &= (u^2+2uv+v^2)-(u^2-2uv+v^2)-4uv\\ &= u^2+2uv+v^2-u^2+2uv-v^2-4uv\\ &= 4uv-4uv\\ &= 0\end{aligned}$$

(b)

The parametrization defined by

$$\mathbf{x}(u,v)=(u\cosh v, u\sinh v, u^2)$$

covers S due to the following.

$$\begin{aligned}(u\cosh v)^2-(u\sinh v)^2-u^2 &= u^2\cosh^2 v-u^2\sinh^2 v-u^2\\ &= u^2(\cosh^2 v-\sinh^2 v)-u^2\\ &= u^2-u^2\\ &= 0\end{aligned}$$

Coverings of the parametrizations ???

5 do Carmo pg 67 exercise 16

(a)

For the π^{-1} definition of

$$\pi^{-1}(u,v)=\begin{cases} x=\frac{4u}{u^2+v^2+4}\\ y=\frac{4v}{u^2+v^2+4}\\ z=\frac{2(u^2+v^2)}{u^2+v^2+4}\end{cases}$$

Letting $d=u^2+v^2$ for convience we have the following.

$$\begin{aligned}x^2+y^2+(z-1)^2 &= x^2+y^2+z^2-2z+1\\ &= \frac{16u^2}{(d+4)^2}+\frac{16v^2}{(d+4)^2}+\frac{4d^2}{(d+4)^2}-\frac{4d}{d+4}+1\\ &= \frac{1}{(d+4)^2}\left(16u^2+16v^2+4d^2-4d(d+4)+(d+4)^2\right)\\ &= \frac{1}{(d+4)^2}\left(16u^2+16v^2+4d^2-4d^2-16d+(d+4)^2\right)\\ &= \frac{1}{(d+4)^2}\left(16u^2+16v^2-16d+(d+4)^2\right)\\ &= \frac{1}{(d+4)^2}\left(16u^2+16v^2-16u^2-16v^2+(d+4)^2\right)\\ &= 1\end{aligned}$$

(b)

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The function $f(x) = x^3$ satisfies our needs. It is certainly smooth. It is bijective with inverse $f^{-1}(x) = \sqrt[3]{x}$ and since both it and the inverse are continuous, then f is a homeomorphism. However, the inverse is not differentiable at $x = 0$ since

$$\frac{d}{dx}\sqrt[3]{x} = \frac{1}{3\sqrt[3]{x^2}}$$

and this is undefined at $x = 0$.

References

[1] Rudin, Walter. *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill Inc. New York, 1976.