Math 501: Differential Geometry
Homework 5

Lawrence Tyler Rush
<me@tylerlogic.com>

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1

Using the formula
$$|u \wedge v|^2 = |u|^2|v|^2 - (u \cdot v)^2$$
we have the following sequence of equations for paramtrization $x : U \to S$ for some regular surface S with $V = x(U)$

$$\begin{aligned}
\text{area}(V) &= \int\int_U |x_u \wedge x_v| dudv \\
&= \int\int_U \sqrt{|x_u \wedge x_v|^2} dudv \\
&= \int\int_U \sqrt{|x_u|^2|x_v|^2 - (x_u \cdot x_v)^2} dudv \\
&= \int\int_U \sqrt{(x_u \cdot x_u)(x_v \cdot x_v) - (x_u \cdot x_v)^2} dudv \\
&= \int\int_U \sqrt{EG - F^2} dudv \\
&= \int\int_U \sqrt{\det(g)} dudv
\end{aligned}$$

where $g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

2

Let S be the graph of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with coordinate mapping $X(u, v) = (u, v, f(u, v))$. From this we get that $X_u = (1, 0, f_u)$ and $X_v = (0, 1, f_v)$.

(a) Coefficients of The First Fundamental Form

$$\begin{aligned}
E &= X_u \cdot X_u = (1, 0, f_u) \cdot (1, 0, f_u) = 1 + f_u^2 \\
F &= X_u \cdot X_v = (1, 0, f_u) \cdot (0, 1, f_v) = f_u f_v \\
G &= X_v \cdot X_v = (0, 1, f_v) \cdot (0, 1, f_v) = 1 + f_v^2
\end{aligned}$$

(b) Length of α

Let $\alpha(t)$ be the curve in S with coordinate expression of (t, t) for $0 \leq t \leq 1$, i.e. $\alpha(t) = (u(t), v(t))$ where $u(t) = t$ and $v(t) = t$. With this, the length of α is given by the following.

$$\begin{aligned}
\int_0^1 \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt &= \int_0^1 \sqrt{(1 + f_u(t,t)^2)1^2 + 2f_u(t,t)f_v(t,t)(1)(1) + (1 + f_v(t,t)^2)1^2} dt \\
&= \int_0^1 \sqrt{2 + f_u(t,t)^2 + 2f_u(t,t)f_v(t,t) + f_v(t,t)^2} dt \\
&= \int_0^1 \sqrt{2 + (f_u(t,t) + f_v(t,t))^2} dt \\
&= \int_0^1 \sqrt{2 + \left(\frac{d}{dt} f(t,t)\right)^2} dt
\end{aligned}$$

(c) Area of V

Let $V = X(U)$ for some bounded open set U of $\mathbb{R}^2$. Using the equation we derived in the first problem, we compute the area of V as follows.

$$\begin{aligned}\int\int_U \sqrt{EG-F^2}dudv &= \int\int_U \sqrt{(1+f_u)(1+f_v)-f_u^2 f_v^2}dudv \\ &= \int\int_U \sqrt{1+f_u+f_v+f_uf_v-f_u^2f_v^2}dudv\end{aligned}$$

3

Define the following parametrizations of the xy-plane in $\mathbb{R}^3$

$$\begin{aligned}X(u,v) &= (u+v, u-v, 0) \\ Y(r,\theta) &= (r\cos\theta, r\sin\theta, 0)\end{aligned}$$

We can then define the change of coordinate function, $h : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h(u,v) = Y^{-1}\circ X(u,v) = Y^{-1}(u+v,u-v,0) = \left(\sqrt{(u+v)^2+(u-v)^2}, \arctan\frac{u-v}{u+v}\right) = \left(\sqrt{2(u^2+v^2)}, \arctan\frac{u-v}{u+v}\right)$$

with inverse

$$h^{-1}(r,\theta) = X^{-1}\circ Y(r,\theta) = X^{-1}(r\cos\theta, r\sin\theta, 0) = \left(\frac{r\cos\theta+r\sin\theta}{2}, \frac{r\cos\theta - r\sin\theta}{2}\right)$$

resulting in the change-of-coordinate functions at the coordinate level of

$$\begin{aligned}r(u,v) &= \sqrt{2(u^2+v^2)} \\ \theta(u,v) &= \arctan\frac{u-v}{u+v} \\ u(r,\theta) &= \frac{r\cos\theta+r\sin\theta}{2} \\ v(r,\theta) &= \frac{r\cos\theta-r\sin\theta}{2}\end{aligned}$$

Computing $X_u = (1,1,0)$ and $X_v = (1,-1,0)$ we can then compute Y_r and Y_θ in terms of X_u and X_v as follows

$$Y_r = X_u\frac{\partial u}{\partial r} + X_v\frac{\partial v}{\partial r} = \frac{\cos\theta+\sin\theta}{2}(1,1,0) + \frac{\cos\theta-\sin\theta}{2}(1,-1,0) = (\cos\theta, \sin\theta, 0)$$

$$Y_\theta = X_u\frac{\partial u}{\partial \theta} + X_v\frac{\partial v}{\partial \theta} = \frac{r}{2}(-\sin\theta+\cos\theta)(1,1,0) + \frac{r}{2}(-\sin\theta-\cos\theta)(1,-1,0) = (-r\sin\theta, r\cos\theta, 0)$$

4

The parametrization of the rotation of the regular plane curve, $(f(v), g(v)$, of the xz-plane with x and z coordinates given by $f(v)$ and $g(v)$, respectively, is

$$X(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

(a)

With the above definition of the parametrization, we have the following differential of X.

$$dX_{(u,v)} = \begin{pmatrix} -f(v)\sin u & f'(v)\cos u \\ f(v)\cos u & f'(v)\sin u \\ 0 & g'(v)\end{pmatrix}$$

Since the curve of rotation is a regular curve then $g'(v) \neq 0$ for all v which tells us that the columns of $dX_{(u,v)}$ are linearly independent and therefore $dX_{(u,v)}$ is injective.

(b)

With the above definition of the parametrization we get

$$X_u = (-f(v)\sin u, f(v)\cos u, 0)$$

and

$$X_v = (f'(v)\cos u, f'(v)\sin u, g'(v))$$

from which we get

$$\begin{aligned}
E &= X_u \cdot X_u = \big((f(v))^2 \sin^2 u, (f(v))^2 \cos^2 u, 0\big) \\
F &= X_u \cdot X_v = (-f(v)f'(v)\sin u \cos u, f(v)f'(v)\sin u \cos u, 0) \\
G &= X_v \cdot X_v = \big((f'(v))^2 \cos^2 u, (f'(v))^2 \sin^2 u, (g'(v))^2\big)
\end{aligned}$$

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Let $\varphi : S_1 \to S_2$ be a local diffeomorphism with S_2 orientable. Then there is some $N_2 : S_2 \to \mathbb{R}^3$ that is a differentiable field of normal unit vectors. Hence $N_1 = N_2 \circ \varphi$ is differentiable in a neighborhood of any $q \in S_1$, but since $\varphi(q) \in S_2$, then $N_2(\varphi(q))$, which is equal to $N_1(q)$, is a unit normal vector. Thus $N_2 : S_1 \to \mathbb{R}^3$ is a differentiable field of unit normal vectors, i.e. S_1 is orientable.