Math 501: Differential Geometry
Homework 6

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1

Let S be a compact surface. Because S is in $\mathbb{R}^3$, then Heine-Borel tells us that the surface is bounded. Let $v \in S^2$. Because S is bounded, then there exists a plane P with unit-normal v that is "far enough away" from S that P and S have no intersection. Thus we can bring P in towards S along the line containing v until it first touches S at some point q. We know that such a point q exists because S is necessarily closed again due to the implications of Heine-Boral since S is compact. Then P will be tangent to S with q being the point of tangency. Since v is a unit-normal of P, then $N(q)$ will be v. Hence N is surjective.

2

Let S be the graph of a smooth function $f(x, y)$.

(a)

Letting $\mathbf{x}(x, y) = (x, y, f(x, y))$ be a coordinate map for S we have

$$\mathbf{x}_x = (1, 0, f_x),\ \mathbf{x}_y = (0, 1, f_y),\ \mathbf{x}_{xx} = (0, 0, f_{xx}),\ \mathbf{x}_{xy} = (0, 0, f_{xy}),\ \text{and } \mathbf{x}_{yy} = (0, 0, f_{yy})$$

Using the normal given by

$$N(x, y) = \frac{\mathbf{x}_x \wedge \mathbf{x}_y}{|\mathbf{x}_x \wedge \mathbf{x}_y|}$$

the formula for the normal is

$$N(x, y) = \frac{(-f_y, -f_x, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

Denoting $(1 + f_x^2 + f_y^2)^{-1/2}$ by c, we then have

$$e = \langle N, \mathbf{x}_{xx} \rangle = cf_{xx},\ f = \langle N, \mathbf{x}_{xy} \rangle = cf_{xy},\ \text{and } g = \langle N, \mathbf{x}_{yy} \rangle = cf_{yy}$$

as well as

$$E = \langle \mathbf{x}_x, \mathbf{x}_x \rangle = 1 + f_x^2,\ F = \langle \mathbf{x}_x, \mathbf{x}_y \rangle = f_x f_y,\ \text{and } G = \langle \mathbf{x}_y, \mathbf{x}_y \rangle = 1 + f_y^2$$

With these formulas we can use the following formula for the Gauss curvature

$$K = \frac{eg - f^2}{EG - F^2}$$

to arrive at

$$\begin{aligned} K &= \frac{c^2 f_{xx} f_{yy} - c^2 (f_{xy})^2}{(1 + f_x^2)(1 + f_y^2) - f_x^2 f_y^2} \\ &= c^2 \frac{f_{xx} f_{yy} - (f_{xy})^2}{1 + f_x^2 + f_y^2 + f_x^2 f_y^2 - f_x^2 f_y^2} \\ &= c^2 \frac{f_{xx} f_{yy} - (f_{xy})^2}{1 + f_x^2 + f_y^2} \\ &= \frac{f_{xx} f_{yy} - (f_{xy})^2}{(1 + f_x^2 + f_y^2)^2} \end{aligned}$$

(b)

For $f(x, y) = x^2 + y^2$ we have

$$f_x = 2x,\ f_y = 2y,\ f_{xx} = 2,\ f_{yy} = 2,\ \text{and } f_{xy} = 0$$

which when using the formula derived in part (a), yields a Gaussian curvature of

$$K = \frac{2(2) - 0^2}{(1 + (2x)^2 + (2y)^2)^2} = \frac{4}{(1 + 4x^2 + 4y^2)^2}$$

which is a nonnegative value that appropaches zero at infinity.

(c)

For $f(x, y) = x^2 - y^2$ we have

$$f_x = 2x,\ f_y = -2y,\ f_{xx} = 2,\ f_{yy} = -2,\ \text{and } f_{xy} = 0$$

which when using the formula derived in part (a), yields a Gaussian curvature of

$$K = \frac{2(-2) - 0^2}{(1 + (2x)^2 + (-2y)^2)^2} = \frac{-4}{(1 + 4x^2 + 4y^2)^2}$$

which is a nonpositive value that again appropaches zero at infinity.

(d)

For $f(x, y) = \sqrt{x^2 + y^2}$ we have

$$f_x = x(x^2 + y^2)^{-1/2},\ f_y = y(x^2 + y^2)^{-1/2},\ f_{xx} = (x^2 + y^2)^{-1/2} + x^2(x^2 + y^2)^{-3/2}$$

and

$$f_{yy} = (x^2 + y^2)^{-1/2} + y^2(x^2 + y^2)^{-3/2},\ f_{xy} = xy(x^2 + y^2)^{-3/2}$$

which when using the formula derived in part (a), yields the following Gaussian curvature denoting $(x^2 + y^2)^{-1/2}$ by c

$$K = \frac{(c - x^2c^3)(c - y^2c^3) - (xyc^3)^2}{(1 + (xc)^2 + (yc)^2)^2} = \frac{c^2 - x^2c^4 - y^2c^4 + x^2y^2c^6 - x^2y^2c^6}{(1 + x^2c^2 + y^2c^2)^2} = \frac{c^2 - x^2c^4 - y^2c^4}{(1 + x^2c^2 + y^2c^2)^2}$$

Multiplying the numerator and denominator by c^{-4} results in a numerator for K of

$$c^{-2} - x^2 - y^2$$

and since $c^{-2} = x^2 + y^2$, then K is identically zero.

(e)

If f is only a function of x, then $\mathbf{x}_x = (1, 0, f_x)$, $\mathbf{x}_y = (0, 1, 0)$, $\mathbf{x}_{xx} = (0, 0, 0)$, and $\mathbf{x}_{xy} = (0, 0, 0)$. Due to the latter two, $f = \langle N, \mathbf{x}_{xy} \rangle = 0$ and $g = \langle N, \mathbf{x}_{xx} \rangle = 0$, which makes for a zero Gaussian curvature.

3

Using the following formula for the mean curvature

$$H = \frac{1}{2}\frac{eG - 2fF + Eg}{EG - F^2}$$

and the formulas for e, f, g, E, F and G from the previous problem to get the following; again we denote $(1 + f_x^2 + f_y^2)^{-1}$ by c

$$\begin{aligned} H &= \frac{1}{2}\frac{cf_{xx}(1+f_y^2) - 2cf_{xy}f_xf_y + (1+f_x^2)cf_{yy}}{1+f_x^2+f_y^2} \\ H &= \frac{c}{2}\frac{f_{xx}(1+f_y^2) - 2f_{xy}f_xf_y + (1+f_x^2)f_{yy}}{1+f_x^2+f_y^2} \\ H &= \frac{f_{xx}(1+f_y^2) - 2f_{xy}f_xf_y + (1+f_x^2)f_{yy}}{2(1+f_x^2+f_y^2)^{3/2}} \end{aligned}$$

4

Let S_1 and S_2 be surfaces of the graphs of functions $z = f_1(x, y)$ and $z = f_2(x, y)$, respectively, such that both pass through the origin and are tangent to the $z = 0$ plane. Assume that $f_2(x, y) \geq f_1(x, y) \geq 0$. Because zero is a minimum value for both surfaces, then the all partial derivatives f_{1x}, f_{1y}, f_{2x}, f_{2y} are zero. This then reduces our equations for Gaussian and mean curvature at the origin, derived in problem one, to

$$K_1 = f_{1xx}f_{1yy} - f_{1xy}^2 K_2 = f_{2xx}f_{2yy} - f_{2xy}^2$$

and

$$H_1 = f_{1xx} + f_{1yy} H_2 = f_{2xx} + f_{2yy}$$

Because $f_2(x, y) \geq f_1(x, y) \geq 0$ was assumed, then from Calculus we know that

$$f_{2xx} \geq f_{1xx} \geq 0$$

and

$$f_{2yy} \geq f_{1yy} \geq 0$$

Adding together these two inequalities yields $f_{2xx} + f_{2yy} \geq f_{1xx} + f_{1yy}$ and multiplying them together yields $f_{2yy}f_{2xx} \geq f_{1yy}f_{1xx} \geq 0$. At the origin, the former informs us that $H_2 \geq H_1$ and, upon rotating the xy-plane such that $f_{1xy} = f_{2xy} = 0$, the latter gives us that $K_2 \geq K_1$.

5

Let S be a compact, orientable surface with inward-pointing normal. Thus Heine-Borel tells us that the surface is bounded, so we can enclose it within a sphere. Shrink this sphere until it first touches S and call one of possibly multiple such points p. We know that such points exists since S is closed, again due to Heine-Borel since S is compact. We can now rotate S and the sphere such that p is identified with the origin, $T_p(S)$ is identified with the $z = 0$ plane, and the normal points in the positive direction. Then we can find a neighborhood of p (now identified with zero) such that S is the graph of some function f_S and the sphere is the graph of some function f such that $f_S(x, y) \geq f(x, y) \geq 0$ at the origin. Thus, by the results of the previous problem, the Gaussian curvature of S at p is greater than or equal to that of a sphere. Thus S has strictly positive curvature since a sphere does as well.

6

Let S be a minimal surface.

(a)

Because S is a minimal surface, then for the principle curvatures k_1 and k_2 at any point p, we have the mean curvature is zero, i.e. $\frac{k_1+k_2}{2} = 0$. In other words, $k_1 = -k_2$. Hence the Gaussian curvature k_1k_2 is simply $-k_2^2$, implying that it is always nonpositive.

(b)

The results of problem five and the previous part of this problem imply that there are no compact minimal surfaces in $\mathbb{R}$.