Math 501: Differential Geometry
Homework 8

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1 do Carmo pg 237 problem 1

With $F = 0$, the Christoffel symbols simplify to

$$\begin{aligned}
\Gamma^1_{11} &= \frac{E_u}{2E} \\
\Gamma^2_{11} &= -\frac{E_v}{2G} \\
\Gamma^1_{12} &= \frac{E_v}{2E} \\
\Gamma^2_{12} &= \frac{G_u}{2G} \\
\Gamma^1_{22} &= -\frac{G_u}{2E} \\
\Gamma^2_{22} &= \frac{G_v}{2G}
\end{aligned}$$

From here, one would use the equation

$$-EK = (\Gamma^2_{12})_u - (\Gamma^2_{11})_v + \Gamma^1_{12}\Gamma^2_{11} + \Gamma^2_{12}\Gamma^2_{12} - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{11}\Gamma^2_{12}$$

and expand/simplify appropriately to get the desired answer. However, I was not able to find the correct sequence of expansions/simplifications of the equation resulting from the combination of the above equations.

2 do Carmo pg 237 problem 2

For $E = G = \lambda(u, v)$ and $F = 0$ we can use the previous problem to find K to be

$$K = \frac{-1}{2\sqrt{\lambda\lambda}}\left(\left(\frac{\lambda_v}{\sqrt{\lambda\lambda}}\right)_v + \left(\frac{\lambda_u}{\sqrt{\lambda\lambda}}\right)_u\right) = \frac{-1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right)$$

Thus since $\Delta \log \lambda = \left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u$ then we have that

$$K = -\frac{1}{2\lambda}\Delta(\log \lambda)$$

Now let $\lambda = (u^2 + v^2 + c)^{-2}$. To make the following computation easier to follow, we'll define $\gamma = u^2 + v^2 + c$, i.e. $\lambda = \gamma^{-2}$. We have the following value of K

$$\begin{aligned}
K &= -\frac{1}{2\lambda}\Delta(\log \lambda) \\
&= \frac{-1}{2\lambda}\left(\left(\frac{\lambda_v}{\lambda}\right)_v + \left(\frac{\lambda_u}{\lambda}\right)_u\right) \\
&= \frac{-1}{2}\gamma^2\left(\left(\frac{-2\gamma^{-3}\gamma_v}{\gamma^{-2}}\right)_v + \left(\frac{-2\gamma^{-3}\gamma_u}{\gamma^{-2}}\right)_u\right) \\
&= \frac{-1}{2}\gamma^2\left(\left(-2\gamma^{-1}\gamma_v\right)_v + \left(-2\gamma^{-1}\gamma_u\right)_u\right) \\
&= \gamma^2\left(\left(\gamma^{-1}\gamma_v\right)_v + \left(\gamma^{-1}\gamma_u\right)_u\right) \\
&= \gamma^2\left((-1)\gamma^{-2}\gamma_v^2 + \gamma^{-1}\gamma_{vv} + (-1)\gamma^{-2}\gamma_u^2 + \gamma^{-1}\gamma_{uu}\right) \\
&= -\gamma_v^2 + \gamma\gamma_{vv} - \gamma_u^2 + \gamma\gamma_{uu}
\end{aligned}$$

With our definition of γ, $\gamma_u = 4u$, $\gamma_v = 4v$, and $\gamma_{uu} = \gamma_{vv} = 4$. Continuing the equation above we obtain

$$\begin{aligned}
K &= -\gamma_v^2 + \gamma\gamma_{vv} - \gamma_u^2 + \gamma\gamma_{uu} \\
&= -4v^2 + (u^2 + v^2 + c)(2) - 4u^2 + (u^2 + v^2 + c)(2) \\
&= 4c
\end{aligned}$$

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Let S be a surface with geodesic coordinates X so that $E = 1$ and $F = 0$.

(a) Christoffel Symbols

Since $E = 1$ then $E_u = E_v = 0$ and thus with $F = 0$ most of the Christoffel symbols are zero. Making use of the equations in (2) on pg 232 of do Carmo, we have

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{12} = 0$$

as well as

$$\Gamma^2_{12} = \frac{G_u}{2G},\ \Gamma^1_{22} = \frac{-G_u}{2},\ \text{and } \Gamma^2_{22} = \frac{G_v}{G}$$

(b) Gaussian Curvature

From the equations of the previous part of the problem, due to most of the Christoffel Symbols being zero, we have this small equation for the Gaussian curvature.

$$\begin{aligned}
-K &= \left(\Gamma^2_{12}\right)_u + \left(\Gamma^2_{12}\right)^2 \\
&= \left(\frac{G_u}{2G}\right)_u + \left(\frac{G_u}{2G}\right)^2 \\
&= \frac{1}{2}\left(\frac{G_u}{G}\right)_u + \frac{(G_u)^2}{4G^2} \\
&= \frac{1}{2}\left(\frac{G_{uu}}{G} - \frac{G_u^2}{G^2}\right) + \frac{(G_u)^2}{4G^2} \\
&= \frac{G_{uu}}{2G} - \frac{(G_u)^2}{4G^2} \\
K &= -\frac{G_{uu}}{2G} + \frac{(G_u)^2}{4G^2}
\end{aligned}$$

(c) g for $\sqrt{G}$

Let $g = \sqrt{G}$. Then $G = g^2$, $G_u = 2gg_u$ and $G_{uu} = 2g_u^2 + 2gg_{uu}$, so substituting these equations into the result of the previous part of the problem gets us

$$\begin{aligned}
K &= -\frac{2g_u^2 + 2gg_{uu}}{2g^2} + \frac{4g^2g_u^2}{4g^4} \\
&= -\frac{g_u^2 + gg_{uu}}{g^2} + \frac{g_u^2}{g^2} \\
&= \frac{g_u^2 - g_u^2 + gg_{uu}}{g^2} \\
&= \frac{g_{uu}}{g}
\end{aligned}$$

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(a)

$$X_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$$
$$X_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$E = \langle X_\phi, X_\phi\rangle = \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi = \sin^2\theta$$
$$F = \langle X_\phi, X_\theta\rangle = -\sin\theta\sin\phi\cos\theta\cos\phi + \sin\theta\cos\phi\cos\theta\sin\phi = 0$$
$$G = \langle X_\theta, X_\theta\rangle = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta = \cos^2\theta + \sin^2\theta = 1$$

$$\begin{array}{ll} E_\theta = 2\sin\theta\cos\theta & E_\phi = 0 \\ F_\theta = 0 & F_\phi = 0 \\ G_\theta = 0 & G_\phi = 0 \end{array}$$

$$\begin{array}{ll} \Gamma_{11}^1 \sin^2\theta = \frac{1}{2} 2\sin\theta\cos\theta & \Gamma_{11}^2 = 0 \\ \Gamma_{11}^1 = \cot\theta & \end{array}$$

$$\begin{array}{ll} \Gamma_{12}\sin^2\theta = 0 & \Gamma_{12}^2 = 0 \\ \Gamma_{12} = 0 & \end{array}$$

$$\Gamma_{22}^1 = 0 \quad \Gamma_{22}^2 = 0$$

(b)

Let $W(s) = a(s)X_\theta + b(s)X_\phi$. With

$$\alpha(s) = \left(\sin\theta\cos\left(\frac{s}{\sin\theta}\right), \sin\theta\sin\left(\frac{s}{\sin\theta}\right), \cos\theta\right)$$

we have $u' = -\sin\left(\frac{s}{\sin\theta}\right)$. Thus by equation (1) on page 239 of do Carmo, we have the following since all Christoffel Symbols are zero except for Γ_{11}^1

$$(a' + \Gamma_{11}^1 a u')X_\theta + b'X_\phi = \left(a' - a\cot\theta\sin\left(\frac{s}{\sin\theta}\right)\right) X_\theta + b'X_\phi$$

(c)

Solving the ODEs of

$$\begin{aligned} a' &= a\left(1 + \cot\theta\sin\left(\frac{s}{\sin\theta}\right)\right) \\ b' &= b \end{aligned}$$

we find that

$$b = e^s + b(0)$$

and

$$a = e^{s - \sin\theta\cot\theta\cos\left(\frac{s}{\sin\theta}\right)} + a(0)$$

How much does the parallel transport of a vector rotate after one loop? ???

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With the definition of α above we get

$$\alpha'(s) = \left(-\sin\left(\frac{s}{\sin\theta}\right), \cos\left(\frac{s}{\sin\theta}\right), 0\right)$$
$$\alpha'(s) = \frac{1}{\sin\theta}\left(-\cos\left(\frac{s}{\sin\theta}\right), -\sin\left(\frac{s}{\sin\theta}\right), 0\right)$$

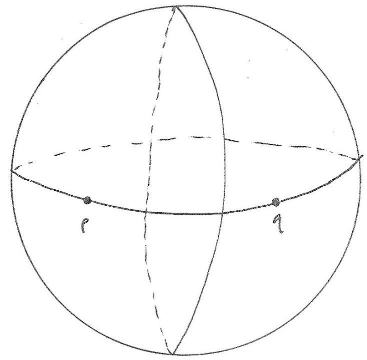

Figure 1: Sketch of positive curvature surface with a geodesic between p and q which is not length-minimizing.

Curvature

$$k = |\alpha'(s)| = \frac{1}{\sin^2\theta}\left(\cos^2\left(\frac{s}{\sin\theta}\right) + \sin^2\left(\frac{s}{\sin\theta}\right)\right) = \frac{1}{\sin^2\theta}$$

Normal Curvature Let N be the normal map on the sphere $N(p) = p$. Than at some p in the trace of α we have

$$\begin{aligned} k_n &= k\,\langle n, N\rangle \\ &= k\left\langle \frac{\alpha''(s)}{k}, \alpha(s)\right\rangle \\ &= \langle \alpha''(s), \alpha(s)\rangle \\ &= -\sin\theta\cos^2\left(\frac{s}{\sin\theta}\right) - \sin\theta\sin^2\left(\frac{s}{\sin\theta}\right) \\ &= -\sin\theta\left(\cos^2\left(\frac{s}{\sin\theta}\right) - \sin^2\left(\frac{s}{\sin\theta}\right)\right) \\ &= -\sin\theta \end{aligned}$$

Geodesic Curvature We have

$$k_g^2 = k^2 - k_n^2 = \sin^2\theta - \sin^4\theta = \sin^2\theta\left(1 - \sin^2\theta\right) = \sin^2\theta\cos^2\theta$$

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(a) Non-length-minimizing positive curvature geodesic

The sketch in Figure 1 has a geodesic between p and q that is not length minimizing. The curve that travels “the long way” along the great circle containing p and q will be a geodesic, but not length-minimizing. The geodesic traveling “the short way” will be length-minimizing though.

(b) Non-length-minimizing zero curvature geodesic

The sketch in Figure 2 has a geodesic between p and q that is not length minimizing. Again going the long way, but this time along a horizontal circle of this cylinder.

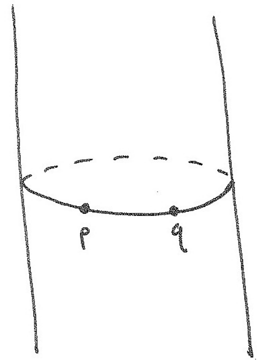

Figure 2: Sketch of zero curvature surface with a geodesic between p and q which is not length-minimizing.

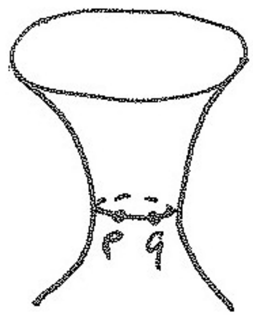

Figure 3: Sketch of negative curvature surface with a geodesic between p and q which is not length-minimizing.

(c) Non-length-minimizing negative curvature geodesic

The sketch in Figure 3 has a geodesic between p and q that is not length minimizing. Again going the long way, but this time around the "waist" of this surface of revolution.

(d)

(e) A surface where any two points can be joined by a geodesic, but the geodesic is only defined for a finite amount of time.

The sketch in Figure 4 has a geodesic between every two points, namely the straight line between them. However, since geodesics need to have constant velocity, then only finite time geodesics could exist.

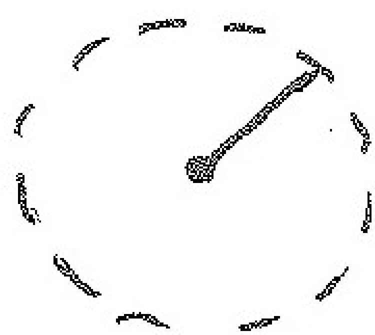

Figure 4: Sketch of surface with only finite time geodesics.