Math 502: Abstract Algebra
Homework 5

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1

Let V be a vector space over a field F and W_1, W_2 be F-vector subspaces of V.

(a) Show $\dim_F \left({}^{V}\!/_{W_i} \right) < \infty$ implies $\dim_F \left({}^{V}\!/_{W_1 \cap W_2} \right) < \infty$

Lemma 1.1. *Let V be a vector space over a field F with subspace W. Then the F-linear transformation $T : V \to {}^{V}\!/_{W}$ defined by $v \mapsto [v]$ is a surjection with kernel W. Furthermore, $\dim {}^{V}\!/_{W} = \dim V - \dim W$.*

Proof. From a group theoretic point of view, $(V, +)$ is an abelian group, and therefore we know T to be the surjective canonical map from the group V to the cosets of its normal subgroup W with kernel W. Thus the `rank+nullity` Theorem yields $\dim V = \dim \ker T + \dim T(V) = \dim W + \dim {}^{V}\!/_{W}$, i.e. $\dim {}^{V}\!/_{W} = \dim V - \dim W$. $\square$

First note that if V is finite dimensional, then this problem is trivial. So assume that V is infinite-dimensional. Then Lemma 1.1 informs us that

$$\dim V - \dim W_i = \dim {}^{V}\!/_{W_i} < \infty \tag{1.1}$$

for each W_i. Noting that

$$\dim V - \dim W_2 < \infty \implies \dim(W_1 + W_2) - \dim W_2 < \infty$$

since $W_2 \subset W_1 + W_2 \subset V$, we then have that the right hand side of the last line of

$$\begin{aligned} \dim V - \dim(W_1 \cap W_2) &= \dim V - (\dim W_1 + \dim W_2 - \dim(W_1 + W_2)) \\ &= (\dim V - \dim W_1) + (\dim(W_1 + W_2) - \dim W_2) \end{aligned}$$

is the sum of two finite numbers. Hence $\dim V - \dim(W_1 \cap W_2)$ is finite, but according to Lemma 1.1 this is the dimension of ${}^{V}\!/_{(W_1 \cap W_2)}$.

(b)

Analogous Statement: If G is a group with $H_1, H_2 \leq G$ such that the index of H_i in G is finite for each H_i, then the index of $H_1 \cap H_2$ in G is also finite.

Let G be a group with subgroups of finite index H_1 and H_2. Then we know that the values of

$${}^{|G|}\!/_{|H_1|} \text{ and } {}^{|G|}\!/_{|H_2|}$$

are both finite. Since $|H_1 \cup H_2| \geq |H_i|$ from a set-theoretic perspective, then we also know ${}^{|G|}\!/_{|H_1 \cup H_2|} = n$ for some finite n. This yields

$$\begin{aligned} \frac{|G|}{|H_1 \cup H_2|} &= n \\ \frac{|G|}{|H_1| + |H_2| - |H_1 \cap H_2|} &= n \\ \frac{|H_1| + |H_2| - |H_1 \cap H_2|}{|G|} &= {}^{1}\!/_{n} \\ \frac{|H_1|}{|G|} + \frac{|H_2|}{|G|} - \frac{|H_1 \cap H_2|}{|G|} &= {}^{1}\!/_{n} \\ \frac{|H_1|}{|G|} + \frac{|H_2|}{|G|} &= {}^{1}\!/_{n} + \frac{|H_1 \cap H_2|}{|G|} \end{aligned}$$

Since the left hand side of the above equation is finite, then so must be the right. Hence $\frac{|H_1 \cap H_2|}{|G|}$ must be finite, and therefore $H_1 \cap H_2$ has finite index in G.

(c) Extra Credit

(d) Extra Credit

2

Let R be a ring and G a group.

(a) Show that the group ring $R[G]$ is a ring.

Has Zero Element The 0 here is that of the ring R is

$$\left(\sum_{x\in G} a_x[x]\right) + \left(\sum_{x\in G} 0[x]\right) = \sum_{x\in G} (a_x + 0)\,[x] = \sum_{x\in G} a_x[x]$$

$$\left(\sum_{x\in G} 0[x]\right) + \left(\sum_{x\in G} a_x[x]\right) = \sum_{x\in G} (0 + a_x)\,[x] = \sum_{x\in G} a_x[x]$$

Addition is associative We get the following because of the associativity of addition on R.

$$\begin{aligned}
\left(\sum_{x\in G} a_x[x] + \sum_{x\in G} b_x[x]\right) + \sum_{x\in G} c_x[x] &= \sum_{x\in G} (a_x + b_x)\,[x] + \left(\sum_{x\in G} c_x[x]\right) \\
&= \sum_{x\in G} ((a_x + b_x) + c_x)\,[x] \\
&= \sum_{x\in G} (a_x + (b_x + c_x))\,[x] \\
&= \sum_{x\in G} a_x[x] + \sum_{x\in G} (b_x + c_x)\,[x] \\
&= \sum_{x\in G} a_x[x] + \left(\sum_{x\in G} b_x[x] + \sum_{x\in G} c_x[x]\right)
\end{aligned}$$

Addition is commutative We get the following by the commutative property of addition on R.

$$\left(\sum_{x\in G} a_x[x]\right) + \left(\sum_{x\in G} b_x[x]\right) = \sum_{x\in G} (a_x + b_x)\,[x] = \sum_{x\in G} (b_x + a_x)\,[x] = \left(\sum_{x\in G} b_x[x]\right) + \left(\sum_{x\in G} a_x[x]\right)$$

Additive elements have inverses The 0 here is that of the ring R is

$$\left(\sum_{x\in G} a_x[x]\right) + \left(\sum_{x\in G} -a_x[x]\right) = \sum_{x\in G} (a_x - a_x)\,[x] = \sum_{x\in G} (0)\,[x] = \left(\sum_{x\in G} 0[x]\right)$$

$$\left(\sum_{x\in G} -a_x[x]\right) + \left(\sum_{x\in G} a_x[x]\right) = \sum_{x\in G} (-a_x + a_x)\,[x] = \sum_{x\in G} (0)\,[x] = \left(\sum_{x\in G} 0[x]\right)$$

Multiplicative Identity The 1 here is that of the ring R is

$$\left(\sum_{x\in G} a_x[x]\right)(1[e]) = \sum_{z\in G}\left(\sum_{x,y\in G,\ xe=z} a_x 1\right)[z] = \sum_{z\in G}(a_z)[z] = \sum_{x\in G} a_x[x]$$

Multiplication is associative We get the following by the associative property of multiplication in R.

$$\begin{aligned}
\left(\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x]\right)\right)\left(\sum_{x\in G} c_x[x]\right) &= \left(\sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} a_x b_y\right)[z]\right)\left(\sum_{x\in G} c_x[x]\right)\\
&= \sum_{w\in G}\left(\sum_{z,t\in G,\ zt=w}\left(\sum_{x,y\in G,\ xy=z} a_x b_y\right)c_t\right)[w]\\
&= \sum_{w\in G}\left(\sum_{z,t\in G,\ zt=w}\sum_{x,y\in G,\ xy=z} a_x b_y c_t\right)[w]\\
&= \sum_{w\in G}\left(\sum_{x,z\in G,\ xz=w}\sum_{y,t\in G,\ yt=z} a_x b_y c_t\right)[w]\\
&= \sum_{w\in G}\left(\sum_{x,z\in G,\ xz=w} a_x\left(\sum_{y,t\in G,\ yt=z} b_y c_t\right)\right)[w]\\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{z\in G}\left(\sum_{y,t\in G,\ yt=z} b_y c_t\right)[z]\right)\\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\left(\sum_{x\in G} b_x[x]\right)\left(\sum_{x\in G} c_x[x]\right)\right)
\end{aligned}$$

Multiplication distributes over addition

$$\begin{aligned}
\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x] + \sum_{x\in G} c_x[x]\right) &= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G}(b_x+c_x)[x]\right)\\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} a_x(b_y+c_y)\right)[z]\\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} a_x b_y + a_x c_y\right)[z]\\
&= \sum_{z\in G}\left(\left(\sum_{x,y\in G,\ xy=z} a_x b_y\right) + \left(\sum_{x,y\in G,\ xy=z} a_x c_y\right)\right)[z]\\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} a_x b_y\right)[z] + \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} a_x c_y\right)[z]\\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x]\right) + \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} c_x[x]\right)
\end{aligned}$$

$$
\begin{aligned}
\left(\sum_{x\in G} b_x[x] + \sum_{x\in G} c_x[x]\right)\left(\sum_{x\in G} a_x[x]\right) &= \left(\sum_{x\in G}(b_x+c_x)[x]\right)\left(\sum_{x\in G} a_x[x]\right) \\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z}(b_x+c_x)a_y\right)[z] \\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} b_xa_y + c_xa_y\right)[z] \\
&= \sum_{z\in G}\left(\left(\sum_{x,y\in G,\ xy=z} b_xa_y\right) + \left(\sum_{x,y\in G,\ xy=z} c_xa_y\right)\right)[z] \\
&= \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} b_xa_y\right)[z] + \sum_{z\in G}\left(\sum_{x,y\in G,\ xy=z} c_xa_y\right)[z] \\
&= \left(\sum_{x\in G} b_x[x]\right)\left(\sum_{x\in G} a_x[x]\right) + \left(\sum_{x\in G} c_x[x]\right)\left(\sum_{x\in G} a_x[x]\right)
\end{aligned}
$$

(b)

(c)

(d)

(e) Extra Credit

(f) Extra Credit

3

(a)

Let $i : D_3 \to \mathrm{GL}_{\mathbb{R}}(\mathbb{C})$ be the inclusion map. Then define $\beta : \mathbb{R}[D_3] \to \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ by

$$
\sum_{s\in D_3} x_s[s] \mapsto \left(z \mapsto \sum_{s\in D_3} x_s(i(s)(z))\right)
$$

Outputs of $\boldsymbol{\beta}$ are indeed in $\mathbf{End}_{\mathbb{R}}(\mathbb{C})$ Let $a, b \in \mathbb{R}$ and $z, z' \in \mathbb{C}$

$$\begin{aligned}
\beta\left(\sum_{s\in D_3} x_s[s]\right)(az+bz') &= \sum_{s\in D_3} x_s\left(i(s)(az+bz')\right) \\
&= \sum_{s\in D_3} x_s(ai(s)(z)+bi(s)(z')) \\
&= \sum_{s\in D_3} (x_s ai(s)(z)+x_s bi(s)(z')) \\
&= \sum_{s\in D_3} x_s ai(s)(z) + \sum_{s\in D_3} x_s bi(s)(z') \\
&= a\sum_{s\in D_3} x_s i(s)(z) + b\sum_{s\in D_3} x_s i(s)(z') \\
&= a\beta\left(\sum_{s\in D_3} x_s[s]\right)(z) + b\beta\left(\sum_{s\in D_3} x_s[s]\right)(z')
\end{aligned}$$

$\boldsymbol{\beta}$ is a homomorphism over addition

$$\begin{aligned}
\beta\left(\sum_{s\in D_3} x_s[s] + \sum_{s\in D_3} y_s[s]\right)(z) &= \beta\left(\sum_{s\in D_3} (x_s+y_s)[s]\right)(z) \\
&= \sum_{s\in D_3} (x_s+y_s)(i(s)(z)) \\
&= \sum_{s\in D_3} (x_s(i(s)(z)) + y_s(i(s)(z))) \\
&= \sum_{s\in D_3} x_s(i(s)(z)) + \sum_{s\in D_3} y_s(i(s)(z)) \\
&= \beta\left(\sum_{s\in D_3} x_s[s]\right)(z) + \beta\left(\sum_{s\in D_3} y_s[s]\right)(z)
\end{aligned}$$

β is a homomorphism over multiplication

$$\begin{aligned}
\beta\left(\left(\sum_{s\in D_3} x_s[s]\right)\left(\sum_{s\in D_3} y_s[s]\right)\right)(z) &= \beta\left(\sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s y_t\right)[u]\right)(z) \\
&= \sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s y_t\right) i(u)(z) \\
&= \sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s y_t i(u)(z)\right) \\
&= \sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s y_t i(st)(z)\right) \\
&= \sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s y_t i(s) i(t)(z)\right) \\
&= \sum_{u\in D_3}\left(\sum_{s,t\in G,\ st=u} x_s i(s)(y_t i(t)(z))\right) \\
&= \sum_{s\in D_3}\left(\sum_{t,u\in G,\ t^{-1}u=s} x_s i(s)(y_t i(t)(z))\right) \\
&= \sum_{s\in D_3} x_s i(s)\left(\sum_{t,u\in G,\ t^{-1}u=s} y_t i(t)(z)\right) \\
&= \beta\left(\sum_{s\in D_3} x_s[s]\right)\left(\sum_{s\in G} y_s i(s)(z)\right) \\
&= \beta\left(\sum_{s\in D_3} x_s[s]\right)\left(\beta\left(\sum_{s\in G} y_s[s]\right)(z)\right) \\
&= \left(\beta\left(\sum_{s\in D_3} x_s[s]\right)\beta\left(\sum_{s\in G} y_s[s]\right)\right)(z)
\end{aligned}$$

This shows that β is a ring homomorphism.

Uniqiueness

(b)

(c) Extra Credit

4

Let $\mathbb{H}$ be the ring of Hamiltonian quaternians and $V = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the three-dimensional vector subspace spanned by the vectors $i, j, k \in \mathbb{H}$.

(a) Show $uxu^{-1} \in V$ for all $u \in \mathbb{H}\times$ and $x \in V$

Let $u \in \mathbb{H}^\times$ and $x, y \in V$. Since we have $u0u^{-1} = 0$ and $u(xy)u^{-1} = ux(u^{-1}u)yu^{-1} = (uxu^{-1})(uyu^{-1})$, then $\mathbb{H}^\times$ acts on V by $u \cdot x = uxu^{-1}$. This induces a homomorphism $\varphi : \mathbb{H}^\times \to \mathrm{Perm}(V)$ by $\varphi(u) = (x \mapsto uxu^{-1})$. Hence, since the range of φ is $\mathrm{Perm}(V)$, we have $uxu^{-1} \in V$ for all $u \in \mathbb{H}^\times$.

(b) Show that $\alpha : \mathbb{H}^\times \to \mathrm{GL}_{\mathbb{R}}(V)$ defined by $\alpha(u)(v) = uvu^{-1}$ is a group homomorphism

The map $\alpha : \mathbb{H}^\times \to \mathrm{GL}_{\mathbb{R}}(V)$ defined by $\alpha(u)(v) = uvu^{-1}$ is simply the permutation representation of the action $u \cdot x = uxu^{-1}$, above. This is induced by the action, and is always a homomorphism.

(c) Determine the kernel of α

The $\ker\alpha$ is the set $\{u \in \mathbb{H}^\times | \alpha(u) = \mathrm{id}_V\}$; in other words it's the set $\{u \in \mathbb{H}^\times \mid uvu^{-1} = v,\ \forall v \in V\}$. Let $Z \subset \mathbb{H}^\times$ be the set of all invertible elements that commute with every element of $\mathbb{H}$. Then it's clear that $Z \subset \ker\alpha$ since an element $u \in Z$ that commutes with every element of $\mathbb{H}$, will certainly have $uvu^{-1} = v$ for all $v \in V$.

So now let $u \in \ker\alpha$. Then for any $a + bi + cj + dk \in \mathbb{H}$ we have that $u(a + bi + cj + dk)u^{-1} = uau^{-1} + u(bi + cj + dk)u^{-1} = a + bi + cj + dk$, since $a \in \mathbb{R}$ commutes with all of $\mathbb{H}$. Thus $u \in Z$, implying that $\ker\alpha \subset Z$.

Putting together the results $Z \subset \ker\alpha$ and $\ker\alpha \subset Z$ from above, we get that $\ker\alpha = Z$, i.e $\ker\alpha$ is the set of all units that commute with all elements of $\mathbb{H}$.

(d) Extra Credit

5

For the sake of clarity, we will use $\{0, 1, 2\}$ as the set $\mathbb{F}_3$, with the necessary understanding that the elements aren't really integers.

(a) Show that $\#\,\mathrm{GL}_2(\mathbb{F}_3) = 48$

There are a total of $3^4 = 81$ possible elements in $M_2(\mathbb{F}_3)$ and $\mathrm{GL}_2(\mathbb{F}_3)$ will be the matrices with nonzero determinant, so we'll subtract the number of matrices with determinant zero from 81. The matrices with zero determinant and independent of the non-zero entries are the zero matrix, the matrices with three zero entries

$$\begin{pmatrix} \alpha & \\ & \end{pmatrix} \quad \begin{pmatrix} & \alpha \\ & \end{pmatrix} \quad \begin{pmatrix} & \\ \alpha & \end{pmatrix} \quad \begin{pmatrix} & \\ & \alpha \end{pmatrix}$$

and the matrices with two non-diagonal zero entries

$$\begin{pmatrix} \alpha & \beta \\ & \end{pmatrix} \quad \begin{pmatrix} \alpha & \\ \beta & \end{pmatrix} \quad \begin{pmatrix} & \\ \alpha & \beta \end{pmatrix} \quad \begin{pmatrix} & \alpha \\ & \beta \end{pmatrix}$$

These account for 1, $4(2) = 8$, and $4(2^2) = 16$, respectively, of the matrices with zero determinant. Thus 25 of the matrices with zero determinant. The remaining such matrices are matrices with all non-zero entries where the product of the two diagonal entries are equal. In other words, a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.2}$$

with $a, b, c, d \neq 0$ we'll have zero determinant when $ad = bc$. Therefore the following "truth" table of possible values of $ad - bc$ reveals that there are eight matrices in the form of 5.2 above that have zero determinant.

a	d	b	c	ad	bc	$ad - bc$
1	1	1	1	1	1	0
1	1	1	2	1	2	2
1	1	2	1	1	2	2
1	1	2	2	1	1	0
1	2	1	1	2	1	1
1	2	1	2	2	2	0
1	2	2	1	2	2	0
1	2	2	2	2	1	1
1	1	1	1	1	1	0
1	1	1	2	1	2	2
1	1	2	1	1	2	2
1	1	2	2	1	1	0
1	2	1	1	2	1	1
1	2	1	2	2	2	0
1	2	2	1	2	2	0
1	2	2	2	2	1	1

Thus adding 8 to the 25 we tallied initially, we have a total of 33, resulting in $\# \mathrm{GL}_2(\mathbb{F}_3) = 81 - 33 = 48$.

(b) Find explicitly a 3-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$

Because of the following

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

then $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ is an element of order 3. However, because $\# \mathrm{GL}_2(\mathbb{F}_3) = 48 = 3(2^4)$, then the subgroup generated by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ is a 3-Sylow subgroup. For the remaining parts of the problem, call it P_3.

(c) Determine the normalizer of P_3. How many 3-Sylow subgroups are there?

Let $a, b, c, d \in \mathbb{F}_3$ and set $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)^{-1} \begin{pmatrix} ad - ac - bc & a^2 \\ -c^2 & ac + ad - bc \end{pmatrix}$$

So in order for A to be in the normalizer of P_3, $-c^2$ must be zero, i.e. c must be zero. This yields

$$(ad - bc)^{-1} \begin{pmatrix} ad - ac - bc & a^2 \\ -c^2 & ac + ad - bc \end{pmatrix} = \begin{pmatrix} 1 & ad^{-1} \\ & 1 \end{pmatrix}$$

Finally we can see that for A to be in the normalizer of P_3, ad^{-1} must be 1 or 2, since an element of the normalizer will permute the non-identity elements of P_3. Furthermore, b is free to be anything in $\mathbb{F}_3$. This implies that

$$N_{\mathrm{GL}_2(\mathbb{F}_3)}(P_3) = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\} \bigcup \left\{ \begin{pmatrix} 2 & b \\ & 2 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\} \bigcup \left\{ \begin{pmatrix} 1 & b \\ & 2 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\} \bigcup \left\{ \begin{pmatrix} 2 & b \\ & 1 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\}$$

or for brevity's sake:

$$N_{\mathrm{GL}_2(\mathbb{F}_3)}(P_3) = \left\{ \begin{pmatrix} u & b \\ & v \end{pmatrix} \middle| b \in \mathbb{F}_3,\ u, v \in \mathbb{F}_3^\times \right\}$$

This means that the size of the normalizer of P_3 is $2(2)(3) = 12$. By Sylow's Theorem's, we know the number of 3-Sylow subgroups of $\mathrm{GL}_2(\mathbb{F})$ to be

1. congruent mod 3
2. divide 16
3. is equal the index of the normalizer of the 3-Sylow subgroup

These come by way of the fact that $\# \mathrm{GL}_2(\mathbb{F}) = 48$ and the size of a 3-Sylow subgroup in the case being 3. Hence the number of 3-Sylow subgroups is four as it is the only value satisfying the above three properties.

(d) Extra Credit: Find, explicitly, a 2-Sylow subgroup.

One such subgroup is the subgroup generated by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

The entire subgroup is:

$$\begin{aligned} \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle = & \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & \\ & 2 \end{pmatrix}, \right. \\ & \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} & 2 \\ 2 & \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, \\ & \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} & 2 \\ 1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ 2 & \end{pmatrix}, \\ & \left. \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \right\} \end{aligned}$$

This 2-Sylow subgroup was discovered via trial/error and intuition with the aide of the following sage [S+13] functions.

```
IDENTITY = matrix.identity(GF(3), 2)

def translations(x, group):
  """ Generates all possible translations of group by x """
  unflattened_translations = []
  for g in group:
    xg = x*g
    gx = g*x
    if gx == xg:
      unflattened_translations.append(gx)
    else:
      unflattened_translations.append((gx,xg))
  return flatten(unflattened_translations)

def cyclic_group(x,y):
  """
  Generates a list of elements contained within the cyclic group generated by x
  and y.
  """
  group = [IDENTITY]
  stack = [x,y]
  while len(stack) != 0:
    m = stack.pop()
    translations_by_m = translations(m, group)
    for new_g in translations_by_m:
      if new_g not in group:
        group.append(new_g)
        stack.append(new_g)
  return group
```

(e) Extra Credit

(f)

References

[S+13] W. A. Stein et al. *Sage Mathematics Software (Version 5.11)*. The Sage Development Team, 2013. http://www.sagemath.org.