Math 502: Abstract Algebra
Homework 6

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(a) 2-Sylow subgroup and its normalizer

If we consider the matrices of the form

$$\begin{pmatrix} \pm 1 & \\ & M \end{pmatrix} \tag{1.1}$$

where M is either

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \text{ or } \begin{pmatrix} & a \\ b & \end{pmatrix}$$

with $a, b = \pm 1$, we can see that there are $2(2^2)$ possible values of M, and 2 possible values for the matrix in equation 1.1 for a fixed value of M. Hence there are $2(2(2^2)) = 16$ total matrices in this form. Since this subset of G contains the identity and has that

$$\begin{pmatrix} \pm 1 & \\ & M \end{pmatrix} \begin{pmatrix} \pm 1 & \\ & M' \end{pmatrix} = \begin{pmatrix} \pm 1 & \\ & MM' \end{pmatrix}$$

then this will be a subgroup of G of size 16, i.e. it's a 2-Sylow subgroup.

Normalizer The 2-Sylow subgroup above is not the only one. For instance the set of matrices of the form

$$\begin{pmatrix} M & \\ & \pm 1 \end{pmatrix}$$

where M is as above is the same kind of subgroup, but different from the 2-Sylow subgroup above. So we must have that there are at least 2 2-Sylow subgroups. By Sylow's Theorem, we know that the number of 2-Sylow subgroups of G, is congruent to $1(\text{mod}2)$ and divides 3. In other words there is either 1 or 3. Since we have already found 2, then there must be exactly 3 2-Sylow subgroups.

Since the number of 2-Sylow subgroups is 3, then so is the index of the normalizer of a given 2-Sylow subgroup. In other words, the normalizer of a 2-Sylow subgroup has size 16, but since this is the size of a 2-Sylow subgroup and subgroups are subsets of their normalizers, then the normalizer of a 2-Sylow subgroup in this case is itself.

(b) 3-Sylow subgroup and its normalizer

Since $\#G = 48$ a 3-Sylow subgroup will have three elements. Since the cyclic subgroup

$$\left\langle \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \right\rangle \tag{1.2}$$

is generated by an element of order three, then it is a 3-Sylow subgroup in this case.

Normalizer Sylows Theorems tell us that the number of 3-Sylow subgroups must (regarding this group) divide 16 and be congruent to 1 mod 3. This leaves 1, 4, and 16 as the possible number of 3-Sylow subgroups. If there were 16, then there would be 32 elements of the group with order 3, but this leaves only 16 elements not of order 3. From above we know there are 3 2-Sylow subgroups (which each have 16 elements), but the number of elements contained in all three is more than 16. Hence we can't have 16 3-Sylow subgroups.

Now because equation 1.2 and the subgroup

$$\left\langle \begin{pmatrix} & 1 & \\ & & -1 \\ -1 & & \end{pmatrix} \right\rangle$$

is a 3-Sylow, then there are at least two 3-Sylow subgroups. Hence there must be four total 3-Sylow subgroups. This in turn is the index of the normalizer of the 3-Sylow subgroup from equation 1.2, call it H_3. So $\#N(H_3) = 48/4 = 12$.

It's easy to see that all of the diagonal matrices of G will be contained within $N(H_3)$. This accounts for 8 elements (including the identity) in the normalizer. The two non-identity elements of H_3 account for two more, leaving us with 10 so far. Finally, with a little intuition, we find that

$$
\begin{aligned}
\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} &= \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \\
\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} &= \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} \\
\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} &= \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \\
\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}\begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix} &= \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}
\end{aligned}
$$

implying that

$$
\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \text{ and } \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}
$$

are both elements of $N(H_3)$. This now accounts for all 12 elements of the normalizer. In summary

$$
N(H_3) = H_3 \bigcup \left\{ \begin{pmatrix} & & a \\ & a & \\ a & & \end{pmatrix} \middle| \; a \in \{-1, 1\} \right\} \bigcup \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| \; a, b, c \in \{-1, 1\} \right\}
$$

(c) Extra Credit

We will demonstrate this by letting the vector space V over F simply be the vector space of the group ring $F[G]$. Define $h : F[G] \to \mathrm{End}_F(V)$ by

$$\sum_{x\in G} a_x[x] \mapsto \left(\sum_{x\in G} b_x[x] \mapsto \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x]\right)\right)$$

in other words, h is just left multiplication by it's input.

The range of h is indeed $\mathrm{End}_F(F[G])$ The following shows that outputs of h are F-endomorphisms on $F[G]$, for scalars $\alpha, \beta \in F$. We make use of the distribution law on $F[G]$.

$$\begin{aligned}
h\left(\sum_{x\in G} a_x[x]\right)\left(\alpha\sum_{x\in G} b_x[x] + \beta\sum_{x\in G} c_x[x]\right) &= h\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} \alpha b_x[x] + \sum_{x\in G} \beta c_x[x]\right) \\
&= h\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} (\alpha b_x + \beta c_x)[x]\right) \\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} \alpha b_x[x] + \sum_{x\in G} \beta c_x[x]\right) \\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} \alpha b_x[x]\right) + \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} \beta c_x[x]\right) \\
&= \alpha\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x]\right) + \beta\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= \alpha h\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x]\right) + \beta h\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} c_x[x]\right)
\end{aligned}$$

***h* preserves addition** Since $h(x)$ is just left multiplication by $x \in F[G]$, then this simply comes by way of the distribution property on $F[G]$:

$$\begin{aligned}
h\left(\sum_{x\in G} a_x[x] + \sum_{x\in G} b_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) &= h\left(\sum_{x\in G} (a_x + b_x)[x]\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= \left(\sum_{x\in G} (a_x + b_x)[x]\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) + \left(\sum_{x\in G} b_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= h\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) + h\left(\sum_{x\in G} b_x[x]\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= \left(h\left(\sum_{x\in G} a_x[x]\right) + h\left(\sum_{x\in G} b_x[x]\right)\right)\left(\sum_{x\in G} c_x[x]\right)
\end{aligned}$$

***h* preserves multiplication** Since $h(x)$ is just left multiplication by $x \in F[G]$, this comes by way of associativity of multiplication on $F[G]$:

$$\begin{aligned}
h\left(\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x\right)\right)\left(\sum_{x\in G} c_x[x]\right) &= \left(\left(\sum_{x\in G} a_x[x]\right)\left(\sum_{x\in G} b_x[x\right)\right)\left(\sum_{x\in G} c_x[x]\right) \\
&= \left(\sum_{x\in G} a_x[x]\right)\left(\left(\sum_{x\in G} b_x[x\right)\left(\sum_{x\in G} c_x[x]\right)\right) \\
&= \left(\sum_{x\in G} a_x[x]\right)\left(h\left(\sum_{x\in G} b_x[x\right)\left(\sum_{x\in G} c_x[x]\right)\right) \\
&= h\left(\sum_{x\in G} a_x[x]\right)\left(h\left(\sum_{x\in G} b_x[x\right)\left(\sum_{x\in G} c_x[x]\right)\right) \\
&= \left(h\left(\sum_{x\in G} a_x[x]\right)\circ h\left(\sum_{x\in G} b_x[x\right)\right)\left(\sum_{x\in G} c_x[x]\right)
\end{aligned}$$

***h* is injective** Because $F[G]$ is a ring, then zero is the only element that results in zero via left multiplication. Because of this, given the definition of h as left multiplication, then zero in $F[G]$ is the only element in the kernel of h, i.e. h is injective.

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(a) Extra Credit

(b)

(c)

(d)

(e)

(f)

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(a)

Assuming that problem 3f above had been proven, we would them have that any finite abelian group of size $27 = 3^3$ would be isomorphic to exactly one of $\mathbb{Z}/27\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

(b)

Show $\boldsymbol{H(\mathbb{F}_3)}$ is a subgroup of $\mathbf{GL_3(\mathbb{F})}$ The subset $H(\mathbb{F}_3)$ contains the identity of $\mathrm{GL}_3(\mathbb{F}_3)$. The set is closed under matrix multiplication by the following

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x & 2z + xy \\ & 1 & 2y \\ & & 1 \end{pmatrix}$$

and each element of $H(\mathbb{F}_3)$ has an inverse by

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & -z + xy \\ & 1 & -y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x - x & -z + xy - xy + z \\ & 1 & y - y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -x & -z + xy \\ & 1 & -y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x - x & -z + xy - xy + z \\ & 1 & y - y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Show $\boldsymbol{H(\mathbb{F}_3)}$ has 27 elements Since there are three possible values for each of x, y, z in

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

and all elements in $H(\mathbb{F}_3)$ are of this form, then there are $3^3 = 27$ elements in $H(\mathbb{F}_3)$

Show $\boldsymbol{H(\mathbb{F}_3)}$ does not have an element of order 9 There is not element of order nine in $H(\mathbb{F}_3)$ because

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 3x & 3z + 3xy \\ & 1 & 3y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

which means that no element can have order greater than three.

(c) Extra Credit

(d) Extra Credit

5 Extra Credit