Math 502: Abstract Algebra
Homework 7

Lawrence Tyler Rush
<me@tylerlogic.com>

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Let $B(\mathbb{F}_3)$ be the subgroup of $\mathrm{GL}_2(\mathbb{F})$ consisting of upper triangular matrices.

(a) Show that $\#B(\mathbb{F}_3) = 12$

Because the determinant of every element in $B(\mathbb{F}_3)$ needs be nonzero, then the diagonal entries of each element must be nonzero. Hence

$$B(\mathbb{F}_3) = \left\{ \begin{pmatrix} a & x \\ & b \end{pmatrix} \middle| a, b \in \mathbb{F}_3^\times \; x \in \mathbb{F}_3 \right\}$$

Since $\#\mathbb{F}_3^\times = 2$ and $\#\mathbb{F}_3 = 3$, then $\#B(\mathbb{F}_3) = 2^2(3) = 12$.

(b)

It takes a little playing around, but it can be realized that the following

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \begin{pmatrix} a & b \\ & c \end{pmatrix} \tag{1.1}$$

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \begin{pmatrix} & c \\ a & b \end{pmatrix} \tag{1.2}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \begin{pmatrix} a & b+c \\ a & b \end{pmatrix} \tag{1.3}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ & c \end{pmatrix} = \begin{pmatrix} 2a & 2b+c \\ a & b \end{pmatrix} \tag{1.4}$$

where $\begin{pmatrix} a & b \\ & c \end{pmatrix} \in B(\mathbb{F}_3)$ and the fact that $a \in \mathbb{F}_3^\times$ imply that the left-most matrices of the above four equations are each in different elements of $\mathrm{GL}_2(\mathbb{F}_3)/B(\mathbb{F}_3)$. This is due to the fact that the first column of each of the matrices on the righthand side of the equations above are distinct. With this, we will henceforth denote the elements of $\mathrm{GL}_2(\mathbb{F}_3)/B(\mathbb{F}_3)$ by

$$\begin{aligned} B_1 &= B(\mathbb{F}_3) \\ B_2 &= \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} B(\mathbb{F}_3) \\ B_3 &= \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} B(\mathbb{F}_3) \\ B_4 &= \begin{pmatrix} 2 & 1 \\ 1 & \end{pmatrix} B(\mathbb{F}_3) \end{aligned}$$

These four sets will be our elements of the permutation group S_4.

Kernel of ρ There is some information regarding the image of a matrix under ρ that we can obtain. Since the sets $\{B_i\}$ are defined by the form of their first column, we can simply look at the multiplication of the first columns by an arbitrary matrix in $\mathrm{GL}_2(\mathbb{F}_3)$. So given the following mappings of first columns of $B_1, \ldots, B_4$ (respectively) by an arbitrary matrix in $\mathrm{GL}_2(\mathbb{F}_3)$

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a_1 \\ \end{pmatrix} = a_1 \begin{pmatrix} w \\ y \end{pmatrix} \tag{1.5}$$

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} \\ a_2 \end{pmatrix} = a_2 \begin{pmatrix} x \\ z \end{pmatrix} \tag{1.6}$$

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a_3 \\ a_3 \end{pmatrix} = a_3 \begin{pmatrix} w+x \\ y+z \end{pmatrix} \tag{1.7}$$

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a_4 \\ 2a_4 \end{pmatrix} = a_4 \begin{pmatrix} w+2x \\ y+2z \end{pmatrix} \tag{1.8}$$

Thus for ρ to map $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ to the identity of S_4 equations 1.5 and 1.6 imply, respectively, that x and y must both be zero in order to map 1 to itself and 2 to itself. Given that, in order for 3 and 4 to map to themselves, equation 1.7 informs us that $w = z$; and this jives with equation 1.8.
Hence the kernel of ρ is nothing more than the diagonal matrices in $\mathrm{GL}_2(\mathbb{F}_3)$.

Image of ρ ??

(c)

Since the bottom right coordinate of all elements of H have a 1, rather than being "free" like $B(\mathbb{F}_3)$ above, then equations 1.1 through 1.4 inform us that the eight cosets have the following eight forms

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ & 2 \end{pmatrix}$$
$$\begin{pmatrix} & 1 \\ a & b \end{pmatrix}$$
$$\begin{pmatrix} & 2 \\ a & b \end{pmatrix}$$
$$\begin{pmatrix} a & b+1 \\ a & b \end{pmatrix}$$
$$\begin{pmatrix} a & b+2 \\ a & b \end{pmatrix}$$
$$\begin{pmatrix} 2a & 2b+1 \\ a & b \end{pmatrix}$$
$$\begin{pmatrix} 2a & 2b+2 \\ a & b \end{pmatrix}$$

for $a \in \mathbb{F}_3^\times$ and $b \in \mathbb{F}_3$.

Finally, in order for and element $\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ to be contained within the kernel of ξ, the element will (minimally) need to map the first coset above to itself as well as the third coset above to itself. Since

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} aw & bw + x \\ ay & by + z \end{pmatrix}$$

then mapping the first coset above to itself requires that $y = 0$ which in turn implies that $z = 1$. Likewise, because

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} ax & w + bx \\ az & y + bz \end{pmatrix}$$

then mapping the third cosert above to itself requires that $x = 0$ which in turn yields $w = 1$. Therefore $x = y = 0$ and $w = z = 1$, or in other words, the identity is the only member of the kernel of ξ.

(d) Extra Credit

(e) Extra Credit

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(a)

Denote the set of all $F[x]$-module structures, $(F[x], V, \cdot)$ by $\mathcal{F}$.

Let's define $\beta : \mathcal{F} \to \mathrm{End}_F(V)$ by

$$(F[x], V, \cdot) \mapsto (v \mapsto x \cdot v)$$

Because of the distributive and associative properties of the $\cdot$ operation, we know the image of β to indeed be in $\mathrm{End}_F(V)$.

β is surjective If we have a $T \in \mathrm{End}_F(V)$ then we can define $\cdot$ by

$$f(x) \cdot v = f(T)(v)$$

which affords us a $F[x]$-module structure, $(F[x], V, \cdot)$. With this definition, $x \cdot v = T(v)$, and therefore $\beta\left((F[x], V, \cdot)\right) = T$

β is injective Let $(F[x], V, \nu_1)$ and $(F[x], V, \nu_2)$ be elements of $\mathcal{F}$ such that their image under β is equal. This then implies that $(v \mapsto \nu_1(x, v))$ and $(v \mapsto \nu_2(x, v))$ are the same map. This in turn yields that

$$\nu_1(x, v) = \nu_2(x, v) \text{ for all } v \in V \tag{2.9}$$

However, due to the properties required of ν_1 and ν_2 by the module axioms on their respective $F[x]$-modules, equation 2.9 implies that

$$\nu_1(f(x), v) = \nu_2(f(x), v)$$

for all $f(x) \in F[x]$ and $v \in V$. This gives us that $(F[x], V, \nu_1)$ and $(F[x], V, \nu_2)$ are the same modules. Hence the injectivity of β.

(b)

Denote the set of all $F[x_1, \ldots, x_n]$-module structures by $\mathscr{F}$. Also denote, by S, the set

$$\left\{(T_1, \ldots, T_n) \;\middle|\; T_i \in \mathrm{End}_F(V),\ T_iT_j = T_jT_i\ \forall 1 \le i, j \le n\right\}$$

Let's define $\beta : \mathscr{F} \to S$ by

$$(F[x_1, \ldots, x_n], V, \cdot) \mapsto \Big((v \mapsto x_1 \cdot v), \cdots, (v \mapsto x_n \cdot v) \Big)$$

Because of the distributive and associative properties of the $\cdot$ operation, we know that each component of the output of β is indeed in $\mathrm{End}_F(V)$. Furthermore, because

$$(x_i + x_j) \cdot v = x_i \cdot v + x_j \cdot v = x_j \cdot v + x_i \cdot v = (x_j + x_i) \cdot v$$

then we know that each output of β indeed falls in S.

β is surjective Let $(T_1, \ldots, T_n) \in S$. Then we can define $\cdot : F[x_1, \ldots, x_n] \times V \to V$ by

$$f(x_1, \ldots, x_n) \cdot v = f(T_1, \ldots, T_n)(v)$$

which gives $(F[x_1, \ldots, x_n], V, \cdot)$ an $F[x_1, \ldots, x_n]$-module structure. Also we have that $x_i \cdot v = T_i(v)$, which implies that $\beta\big((F[x_1, \ldots, x_n], V, \cdot)\big) = (T_1, \ldots, T_n)$.

$\boldsymbol{\beta}$ is injective Let $(F[x], V, \nu_1)$ and $(F[x], V, \nu_2)$ be elements of $\mathscr{F}$ such that their image under β is equal. This then implies that $(v \mapsto \nu_1(x_i, v))$ and $(v \mapsto \nu_2(x_i, v))$ are the same map for each $1 \le i \le n$. This in turn yields that

$$\nu_1(x_i, v) = \nu_2(x_i, v) \text{ for all } v \in V \tag{2.10}$$

However, due to the properties required of ν_1 and ν_2 by the module axioms on their respective $F[x_1, \ldots, x_n]$-modules, equation 2.10 implies that

$$\nu_1(f(x_1, \ldots, x_n), v) = \nu_2(f(x_1, \ldots, x_n), v)$$

for all $f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ and $v \in V$. This gives us that $(F[x_1, \ldots, x_n], V, \nu_1)$ and $(F[x_1, \ldots, x_n], V, \nu_2)$ are the same modules. Hence the injectivity of β.

(c)

Denote the set of all $F[S_3]$-module structures by $\mathfrak{F}$ and denote, by $\mathfrak{S}$, the set

$$\left\{(S, T) \mid S, T \in \operatorname{End}_F(V) \text{ s.t. } S^2 = \operatorname{Id}_V, T^3 = \operatorname{Id}_V\right\}$$

for some F-vector space V.
Let's define $\beta : \mathfrak{F} \to \mathfrak{S}$ by

$$(F[S_3], V, \cdot) \mapsto \Big((v \mapsto (1,2) \cdot v), \cdots, (v \mapsto (1,2,3) \cdot v) \Big)$$

Because of the distributive and associative properties of the $\cdot$ operation, we know that each component of the output of β is indeed in $\operatorname{End}_F(V)$. Furthermore, because

$$(1,2) \cdot (1,2) \cdot v = ((1,2)(1,2)) \cdot v = (1) \cdot v = v$$

and

$$(1,2,3) \cdot (1,2,3) \cdot (1,2,3) \cdot v = ((1,2,3)(1,2,3)(1,2,3)) \cdot v = (1) \cdot v = v$$

then we know that each output of β indeed falls in $\mathfrak{S}$.

$\boldsymbol{\beta}$ is surjective Let $(S, T) \in \mathfrak{S}$. Then if we define $\cdot : F[S_3] \times V \to V$ by

$$\left(\sum_{(1,2)^i (1,2,3)^j \in S_3} a_{(1,2)^i (1,2,3)^j} [(1,2)^i (1,2,3)^j] \right) \cdot v = \left(\sum_{(1,2)^i (1,2,3)^j \in S_3} a_{(1,2)^i (1,2,3)^j} S^i T^j \right) (v)$$

then $(F[S_3], V, \cdot)$ will have an $F[S_3]$-module structure such that its image under β is (S, T).

$\boldsymbol{\beta}$ is injective Let $(F[S_3], V, \nu_1)$ and $(F[S_3], V, \nu_2)$ be elements of $\mathfrak{F}$ such that their image under β is equal. This then implies that $(v \mapsto \nu_1((1,2), v))$ and $(v \mapsto \nu_2((1,2), v))$ are the same map and $(v \mapsto \nu_1((1,2,3), v))$ and $(v \mapsto \nu_1((1,2,3), v))$ are also the same map. This in turn yields that

$$\nu_1((1,2), v) = \nu_2((1,2), v) \text{ for all } v \in V \tag{2.11}$$

as well as

$$\nu_1((1,2,3), v) = \nu_2((1,2,3), v) \text{ for all } v \in V \tag{2.12}$$

However, due to the properties required of ν_1 and ν_2 by the module axioms on their respective $F[S_3]$-modules, equations 2.11 and 2.12 imply that

$$\nu_1(x, v) = \nu_2(x, v)$$

for all $x \in F[S_3]$ and $v \in V$. This gives us that $(F[S_3], V, \nu_1)$ and $(F[S_3], V, \nu_2)$ are the same modules. Hence the injectivity of β.

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(a)

(b)

(c)

4 Extra Credit