Math 502: Abstract Algebra
Homework 10

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1

(a)

Since T is multiplication by $[\overline{1}]$, then

$$T\left([\overline{0}]\right) = [\overline{1}] \qquad T\left([\overline{1}]\right) = [\overline{2}] \qquad T\left([\overline{2}]\right) = [\overline{3}] \qquad T\left([\overline{3}]\right) = [\overline{0}]$$

which informs us that the matrix of T in the basis $\left\{[\overline{0}], [\overline{1}], [\overline{2}], [\overline{3}]\right\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now given this matrix, we obtain a characteristic polynomial of $x^4 - 1$, which over $\mathbb{C}$ factors as $(x-1)(x+1)(x-i)(x+i)$. Because the minimal and characteristic polynomials share the same roots, then the Cayley-Hamilton Theorem informs us that the minimal and characteristic polynomials are the same in this case. Therefore the elementary divisors are $(x-1)$, $(x+1)$, $(x-i)$, and $(x+i)$, which informs us that the rational canonical form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

(b)

The same beginning argument above applies in this case as well for S, except that the characteristic polynomial factors differently as our field is $\mathbb{Q}$. In this case the characteristic polynomial factors as $x^4-1 = (x-1)(x+1)(x^2+1)$ Again because the minimal and characteristic polynomials share the same roots, then the Cayley-Hamilton Theorem informs us that the minimal and characteristic polynomials are the same in this case. Therefore the elementary divisors are $(x-1)$, $(x+1)$, and (x^2+1), which informs us that the rational canonical form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2

Let V be a vector space of dimension 8 over a field $\mathbb{R}$, and let $T \in \mathrm{End}_{\mathbb{R}}(V)$ such that

$$T^3(T^3-1)^2 = 0 \tag{2.1}$$

We know then that (V, T) corresponds to a finitely generated $\mathbb{R}[x]$-module, and we can therefore apply the structure theorem for finitely generated modules in order to classify all such pairs (V, T).

To do so, we use the information given to us by equation 2.1 to determine the possible minimal polynomials, which we will denote by $m_T(x)$. Equation 2.1 tells us that $m_T(x)$ must divide $x^3(x^3-1)^2$, and therefore $m_T(x)$ could be any of

$$\begin{array}{rrr} x^3(x^3-1)^2 & x^3(x^3-1) & x^3 \\ x^2(x^3-1)^2 & x^2(x^3-1) & x^2 \\ x(x^3-1)^2 & x(x^3-1) & x \\ (x^3-1)^2 & x^3-1 & \end{array}$$

Due to the Cayley-Hamilton theorem, we know the minimal polynomial of T to divide its characteristic polynomial, but the degree of the characteristic polynomial is also bounded above by the dimension of the vector space. Thus we can immediately rule out $x^3(x^3-1)^2$ as a potential value for $m_T(x)$. This of course leaves us with

$$\begin{array}{rrl} & x^3(x^3-1) & x^3 \\ x^2(x^3-1)^2 & x^2(x^3-1) & x^2 \\ x(x^3-1)^2 & x(x^3-1) & x \\ (x^3-1)^2 & x^3-1 & \end{array}$$

as the possible minimal polynomials of T.

Now because the structure theorem for finitely generated modules informs us that (V, T) is identified with

$$\bigoplus_{i=1}^{n} \frac{\mathbb{R}[x]}{a_i(x)}$$

where $a_i(x) \mid a_j(x)$ for each $i < j$. We also know that both $m_T(x) = a_n(x)$ and that the characteristic polynomial is $\prod_{i=1}^{n} a_i(x)$. Thus with all of the above information, and because the degree of the characteristic polynomial is bounded by the dimension of V, i.e. 8, we conclude that (V, T) is completely identified by one of the following:

$$\begin{gathered}
\frac{\mathbb{R}[x]}{\left(x^2(x^3-1)^2\right)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{\left(x(x^3-1)^2\right)} \\
\frac{\mathbb{R}[x]}{(x^2)} \oplus \frac{\mathbb{R}[x]}{\left(x^3(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{\left(x^3(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{x^2} \oplus \frac{\mathbb{R}[x]}{\left(x^2(x^3-1)\right)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 3} \oplus \frac{\mathbb{R}[x]}{\left(x^2(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{(x^3-1)} \oplus \frac{\mathbb{R}[x]}{\left(x^2(x^3-1)\right)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 4} \oplus \frac{\mathbb{R}[x]}{\left(x(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{\left(x(x^3-1)\right)} \oplus \frac{\mathbb{R}[x]}{\left(x(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{(x^3-1)} \oplus \frac{\mathbb{R}[x]}{\left(x(x^3-1)\right)} \\
\frac{\mathbb{R}[x]}{(x^2)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \\
\frac{\mathbb{R}[x]}{(x)} \oplus \frac{\mathbb{R}[x]}{(x^2)} \oplus \frac{\mathbb{R}[x]}{(x^2)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 3} \oplus \frac{\mathbb{R}[x]}{(x^2)} \oplus \frac{\mathbb{R}[x]}{(x^3)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 5} \oplus \frac{\mathbb{R}[x]}{(x^3)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 6} \oplus \frac{\mathbb{R}[x]}{(x^2)} \\
\left(\frac{\mathbb{R}[x]}{(x)}\right)^{\oplus 8}
\end{gathered}$$

3

Denote $\sqrt{-5}$ by ω.

(a) Show that $\mathbb{Z}[\omega] \cong {}^{\mathbb{Z}[x]}/_{(x^2+5)}$

Define the map $\phi : {}^{\mathbb{Z}[x]}/_{(x^2+5)} \to \mathbb{Z}[\omega]$ by $\phi(a\overline{1} + b\overline{x}) = a + b\omega$. This is certainly surjective since we can let both a and b range over $\mathbb{Z}$. It is also injective for if $\phi(a_1\overline{1} + b_1\overline{x}) = \phi(a_2\overline{1} + b_2\overline{x})$ then $a_1 + b_1\omega = a_2 + b_2\omega$ and thus $a_1 = a_2$ and $b_1 = b_2$. Hence we have the bijectivity of ϕ. By the following we have that ϕ is a ring homomorphism.

$$\begin{aligned}
\phi((a\overline{1} + b\overline{x})(c\overline{1} + d\overline{x})) &= \phi(ac\overline{1} + bc\overline{x} + ad\overline{x} + bd\overline{x^2}) \\
&= \phi(ac\overline{1} + (bc + ad)\overline{x} + bd\overline{-5}) \\
&= \phi((ac - 5bd)\overline{1} + (bc + ad)\overline{x}) \\
&= (ac - 5bd) + (bc + ad)\omega \\
&= ac + bd\omega^2 + bc\omega + ad\omega \\
&= ac + bc\omega + ad\omega + bd\omega^2 \\
&= c(a + b\omega) + d\omega(a + b\omega) \\
&= (a + b\omega)(c + d\omega) \\
&= \phi(a\overline{1} + b\overline{x})\phi(c\overline{1} + d\overline{x})
\end{aligned}$$

$$\begin{aligned}
\phi((a\overline{1} + b\overline{x}) + (c\overline{1} + d\overline{x})) &= \phi((a + c)\overline{1} + (b + d)\overline{x}) \\
&= (a + c) + (b + d)\omega \\
&= (a + b\omega) + (c + d\omega) \\
&= \phi(a\overline{1} + b\overline{x}) + \phi(c\overline{1} + d\overline{x})
\end{aligned}$$

In summary, ϕ is and isomorphism.

(b) Extra Credit

(c) Extra Credit

4

(a)

(b) Extra Credit

(c) Extra Credit

5

(a)

Let $g \in G$ and define $T_g \in \mathrm{End}_{\mathbb{C}}(V)$ by $T_g(v) = g \cdot v$. With this definition, since $d = \#G$, then $(T_g)^d(v) = g^d \cdot v = e \cdot v = v$ where e is the identity of G. This implies that $x^d - 1$ annihilates V for T, which further implies that the minimal polynomial divides $x^d - 1$. Since our vector space is over the field $\mathbb{C}$, then $x^d - 1$ factors completely into linear factors. Also, according to the hint that $x^d - 1$ has no multiple roots, we have that $x^d - 1$, factors completely into distinct linear factors, and therefore so does the minimal polynomial. Hence T is semisimple. This, according to problem four of the previous homework, implies that T is diagonalizable, since $\mathbb{C}$ is algebraically closed.

(b)

(c) Extra Credit

(d) Extra Credit

(e) Extra Credit
