Math 502: Abstract Algebra
Homework 12

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In Collaboration With

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Let G be a finite group and $\mathbb{C}(G)$ be the set of all $\mathbb{C}$-valued functions on G. Define an inner-product on the $\mathbb{C}$-vector space by

$$(f_1|f_2) := \frac{1}{\#G}\sum_{x\in G} f_1(x)\cdot\overline{f_2(x)}$$

(a)

For each $g \in G$, let $U_g \in \mathrm{End}_{\mathbb{C}}(C(G))$ be defined by $f \mapsto (x \mapsto f(xg)$. With this definition, we see that

$$\begin{aligned}\big(U_g(f_1)|U_g(f_2)\big) &= \frac{1}{\#G}\sum_{x\in G} U_g(f_1)(x)\cdot\overline{U_g(f_2)(x)}\\ &= \frac{1}{\#G}\sum_{x\in G} f_1(xg)\cdot\overline{f_2(xg)}\end{aligned}$$

but because right translation in G is a bijection, we can alter the index on G to be

$$\begin{aligned}\big(U_g(f_1)|U_g(f_2)\big) &= \frac{1}{\#G}\sum_{y\in G} f_1(y)\cdot\overline{f_2(y)}\\ &= (f_1|f_2)\end{aligned}$$

and hence U_g is a unitary operator for each $g \in G$.

(b)

For each $g \in G$, let $U_g \in \mathrm{End}_{\mathbb{C}}(C(G))$ be defined by $f \mapsto (x \mapsto f(xg)$. With this definition, we see that

$$\begin{aligned}\big(U_g(f_1)|f_2\big) &= \frac{1}{\#G}\sum_{x\in G} U_g(f_1)(x)\cdot f_2(x)\\ &= \frac{1}{\#G}\sum_{x\in G} f_1(xg)\cdot f_2(x)\end{aligned}$$

however in the above equation, we can substitute y for xg to get

$$\big(U_g(f_1)|f_2\big) = \frac{1}{\#G}\sum_{y\in G} f_1(y)\cdot f_2(yg^{-1})$$

which indicates that $U_{g^{-1}}$ will be the hermitian conjugate of U_g.

(c)

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Let V be a finite dimensional vector space over a field F which is either $\mathbb{C}$ or $\mathbb{R}$ with S a linear operator on V.

(a) Prove S being skew-hermitian implies $\exp(S)$ is unitary

Assume that S is skew-hermitian. Let $\lambda \in F$ be an eigenvalue of S with corresponding eigenvector v. Then we have that

$$\exp(S)(v) = \sum_{k=0}^{\infty} \frac{S^k}{k!}(v) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(v) = e^{\lambda}(v)$$

with the rightmost equality coming from the Taylor series expansion of the exponential function. So we see that e^{λ} is an eigenvalue for $\exp(S)$. Now because S was assumed skew-hermitian, then $\lambda \in \sqrt{-1}\mathbb{R}$, implying that $\lambda = ai$ for some $a \in \mathbb{R}$. So e^{ai} is an eigenvalue of $\exp(S)$, but since it has magnitude of 1, then $\exp(S)$ must be unitary.

(b)

Counter-example:
Let

$$A = \begin{pmatrix} i & \\ & \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

then both A and B are period matrices since

$$A^2 = \begin{pmatrix} -1 & \\ & \end{pmatrix} \; A^3 = \begin{pmatrix} -i & \\ & \end{pmatrix} \; A^4 = \begin{pmatrix} 1 & \\ & \end{pmatrix} \; A^5 = \begin{pmatrix} i & \\ & \end{pmatrix}$$

and

$$B^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \; B^3 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \; B^4 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \; B^5 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

Therefore since $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges (to e, although it's immaterial), then $\exp(A) = I + a_1A^1 + a_2A^2 + a_3A^3 + a_4A^4$ and $\exp(B) = b_1B^1 + b_2B^2 + b_3B^3 + b_4B^4$ where each $a_i, b_i \in F$. In other words, $\exp(A) \cdot \exp(B)$ is a finite linear combination of 16 matrices.

Now I'd like to show that $\exp(A + B)$ is not a finite linear combination of matrices, but I am unsure how to do so.

(c) Extra Credit

3 Extra Credit

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(a)

If we let $T \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ be multiplication by $b_0[0] + b_1[1] + \cdots + b_{n-1}[n-1]$, then we have that

$$\begin{aligned}
T([0]) &= b_0[0] + b_1[1] + \cdots + b_{n-1}[n-1] \\
T([1]) &= b_{n-1}[0] + b_0[1] + b_1[2] + \cdots + b_{n-2}[n-1] \\
&\vdots \\
T([n-1]) &= b_1[0] + b_2[1] + \cdots + b_0[n-1]
\end{aligned}$$

which implies that the matrix representation of T in the basis $\{[0], [1], \ldots, [n-1]\}$ of $\mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ is A_n.

(b)

(c) Extra Credit

(d) Extra Credit
