Math 502: Abstract Algebra
Homework 13

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We first lay out a helpful lemma. [1]

Lemma 1.1. *Let V be a vector space over F where F is $\mathbb{R}$ or $\mathbb{C}$. If S and T are hermitian and $ST = TS$, then there is an orthogonal basis for S and T.*

Proof. Let λ be an eigenvalue of T and set W to be the eigenspace with respect to lambda. Then the commutativity of ST and TS gives us that

$$TSw = STw = S(\lambda w) = \lambda(Sw)$$

for $w \in W$. In other words $Sw \in W$, implying that W is S-invariant. Now since T is hermitian, then W is 1-dimensional, implying that *lambda* is an eigenvalue for S and furthermore the eigenspace for S corresponding to λ is W. Since λ is arbitrary and both S and T are diagonalizable, then we know that there exists an orthogonal basis $v_1, \cdots, v_n$. □

Now for the main event. Let V be a vector space over F where F is $\mathbb{R}$ or $\mathbb{C}$. Also let $(\cdot|\cdot)_1$ and $(\cdot|\cdot)_2$ be two distinct inner products on V. Due to Gram-schmitt, we can assume that $(\cdot|\cdot)_1$ is simply the standard inner product on V. So Then there exists hermitian, positive definite operators $T \in \mathrm{End}_F(V)$ such that $(x|y)_2 = (Tx|y)_1$, and in the case of $(\cdot|\cdot)_1$, the hermitian operator corresponding to it is Id. Certainly $T\,\mathrm{Id}$ and $\mathrm{Id}\,T$ are commutative and T and Id are both hermitian. Hence Lemma 1.1 implies that we can find an orthogonal basis for the standard inner product, i.e. $(\cdot|\cdot)_1$, but because $(x|y)_2 = (Tx|y)_1$, then this will be othogonal with respect to both inner products.

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(a)

(b)

Let $w \in \mathrm{Ker}(S - \lambda Id_W)$ then we have that $(S - \lambda Id_W)w = 0$, and so $Sw = Tw = \lambda w$, which implies that $w \in W(\lambda)$.

Now assume that $w \in W(\lambda)$, then $Sw = Tw = \lambda w$, which implies that $w \in \mathrm{Ker}(S - \lambda Id_W)$.

Hence $\mathrm{Ker}(S - \lambda Id_W) = W(\lambda)$.

(c)

We first factor $S^2 - (\lambda - \overline{\lambda})S + \lambda\overline{\lambda}\,\mathrm{Id}_W$.

$$\begin{aligned} S^2 - (\lambda - \overline{\lambda})S + \lambda\overline{\lambda}\,\mathrm{Id}_W &= S^2 - \lambda S - \overline{\lambda}S + \lambda\overline{\lambda}\,\mathrm{Id}_W \\ &= S(S - \lambda\,\mathrm{Id}_W) - \overline{\lambda}(S - \lambda\,\mathrm{Id}_W) \\ &= (S - \overline{\lambda}\,\mathrm{Id}_W)(S - \lambda\,\mathrm{Id}_W) \end{aligned}$$

First assume that $w \in W(\lambda) + W(\overline{\lambda})$, then $w = u + v$ where $u \in W(\lambda)$ and $v \in W(\overline{\lambda})$. But then u and v is killed by $(S - \overline{\lambda}\,\mathrm{Id}_W)$ and $(S - \lambda\,\mathrm{Id}_W)w$, respectively. Hence $u + v = w$ is killed by $(S - \overline{\lambda}\,\mathrm{Id}_W)(S - \lambda\,\mathrm{Id}_W)w$.

Now assume that $w \in \mathrm{Ker}(S^2 - (\lambda - \overline{\lambda})S + \lambda\overline{\lambda}\,\mathrm{Id}_W) = \mathrm{Ker}((S - \overline{\lambda}\,\mathrm{Id}_W)(S - \lambda\,\mathrm{Id}_W))$.

(d)

[1] Many ideas for this proof came from reading [HJE03]

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Define $Q(v)$ as a quadratic form on $\mathbb{R}^n$ by

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i,j \leq n} a_{ij} x_i x_j$$

for some *symmetric* $A = (a_{ij}) \in M_n(\mathbb{R})$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A with possible multiplicity.

(a) Show that $\lambda_1 = \max\{Q(v) \mid v \in \mathbb{R}^n, (v|v) = 1\}$

Because A is symmetric, we can obtain an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ with respect to the dot product corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any $v \in \mathbb{R}^n$ we have $v = b_1 v_1 + \cdots + b_n v_n$ which implies

$$\begin{aligned} Q(v) &= (Av|v) \\ &= (b_1\lambda_1 v_1 + \cdots + b_n\lambda_n v_n)|v) \\ &= (b_1\lambda_1 v_1|v) + \cdots + (b_n\lambda_n v_n|v) \\ &= (b_1\lambda_1 v_1|b_1 v_1) + \cdots + (b_n\lambda_n v_n|b_n v_n) \\ &= b_1^2\lambda_1 + \cdots + b_n^2\lambda_n \end{aligned}$$

with the last two equalities coming to us by way of the orthonormality of β. Now assuming that $(v|v) = 1$, then

$$1 = (v|v) = (b_1 v_1 + \cdots + b_n v_n | b_1 v_1 + \cdots + b_n v_n) = b_1^2 + \cdots + b_n^2$$

again by the orthonormality of β. Hence $\lambda_1 \geq b_1^2\lambda_1 + \cdots + b_n^2\lambda_n$, i.e

$$\lambda_1 = \max\{Q(v) \mid v \in \mathbb{R}^n, (v|v) = 1\}$$

(b) Extra Credit

(c) Extra Credit

4 Extra Credit

5 Extra Credit

References

[HJE03] Friedberg S. H., Insel A. J., and Spence L. E. *Linear Algebra.* Prentice Hall, 2003.