Math 503: Abstract Algebra
Homework 1

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Some Language. Similar to Dummit and Foote [DF04, p. 365], for left R-module N, right R-module M and abelian group L, we will call a map $\alpha : M \times N \to L$ ***R*-balanced** if it satisfies all three of

$$\alpha(m_1 + m_2, n) = \alpha(m_1, n) + \alpha(m_2, n)$$
$$\alpha(m, n_1 + n_2) = \alpha(m, n_1) + \alpha(m, n_2)$$
$$\alpha(mr, n) = \alpha(m, rn)$$

for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

1

Let V and W be modules over a commutative ring R. Then according to the definition of tensor products, we have the existence of balanced maps $\alpha : V \times W \to V \otimes_R W$ and $\alpha' : W \times V \to W \otimes_R V$. Since $V \times W$ and $W \times V$ are isomorphic, we have an isomorphism $i : V \times W \to W \times V$, and so setting $\beta = \alpha' \circ i$ we have that β is a balanced map from $V \times W$ to $W \otimes_R V$. Hence, as $W \otimes_R V$ is an abelian group (it's an R-module), the universal property of the tensor product gives us that there exists a unique homomorphism $s : V \otimes_R W \to W \otimes_R V$ such that the diagram

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & V \otimes_R W \arrow[d, "s"] \\ & W \otimes_R V \end{tikzcd}$$

commutes. Furthermore, since $\alpha(v, w) = v \otimes w$ and $\beta(v, w) = w \otimes v$, then $s(v \otimes w) = w \otimes v$ for each $v \in V$ and $w \in W$.

Finally, since R is commutative, the above argument symmetrically holds when V and W are swapped as each is a left and right R-module. This implies the existence of a homomorphism $s' : W \otimes_R V \to V \otimes_R W$ such that $s'(w \otimes v) = v \otimes w$. Hence, s above is invertible and therefore an isomorphism. [1]

2

(a)

Since under standard scalar multiplication, R_n is a left F-module. Furthermore for $a \in F$, $r \in R_n$ and $A \in M_n(F)$ we have that $a(rA) = (ar)A$ by standard linear algebra rules. Hence R_n is an F-$M_n(F)$-bimodule. This then gives $R_n \otimes_F C_n$ structure as a right F-module, but since F is a field, then $R_n \otimes_F C_n$ is a F-vector space.

(b)

First let us define $\varphi : R_n \times C_n \to F$ by $\varphi(r, c) = rc$ for each $r \in R_n$ and $c \in C_n$. This is essentially the dot product, and so we know it to be bilinear, but just for the sake of completeness:

$$\varphi(rA, c) = (rA)c = r(Ac) = \varphi(r, Ac)$$
$$\varphi(r_1 + r_2, c) = (r_1 + r_2)c = r_1c + r_2c = \varphi(r_1, c) + \varphi(r_2, c)$$
$$\varphi(r, c_1 + c_2) = r(c_1 + c_2) = rc_1 + rc_2 = \varphi(r, c_1) + \varphi(r, c_2)$$

This then implies that φ is $M_n(F)$-balanced. Hence the universal property of tensor products allots us the existence of a unique $\phi : R_n \otimes_{M_n(F)} C_n \to F$ such that $\varphi = \phi \circ i$ where i is the normal "inclusion" map. With this we have two helpful lemmas.

[1] Inspiration for this proof drawn from [Lan02, p. 605]

Lemma 2.1. *Let ϕ be defined as it is above. For all $r \in R_n$ and $c \in C_n$, if $\phi(r \otimes c) = 0$ then $r \otimes c = 0 \in R_n \otimes_{M_n(F)} C_n$.*

Proof. Let the initial conditions of the Lemma's statement stand. Denote the components of r and c by $r_1, \ldots, r_n$ and $c_1, \ldots, c_n$, respectively. Since $\phi(r \otimes c) = rc = 0$ then for each i, $r_i = 0$ or $c_i = 0$. Let $i_1, \ldots, i_k$ be the indices of the components of r which are zero. Therefore $c_j = 0$ for $j \notin \{i_1, \ldots, i_k\}$ Let $A \in M_n(R)$ be the matrix with ones along the diagonal, except in rows $i_1, \ldots, i_k$ which shall be completely zero. Then we have that

$$r \otimes c = rA \otimes c = r \otimes Ac = r \otimes 0 = 0$$

□

Lemma 2.2. *For every element $x \in R_n \otimes_{M_n(F)} C_n$ there exist elements $r \in R_n$ and $c \in C_n$ such that $x = r \otimes c$.*

Proof. As $R_n \otimes_{M_n(F)} C_n$ is made up of *finite* linear combinations of elements of the form $r \otimes c$, it suffices to only show that any two such elements can be combined into one. So let $r_1 \otimes c_1 + \cdots + r_2 \otimes c_2$ be arbitrary in $R_n \otimes_{M_n(F)} C_n$. Then by denoting the individual components of r_1 by $r_{11}, \ldots, r_{1n}$ and likewise for r_2, we can define $A \in M_n(F)$ to be

$$A = \begin{pmatrix} r_{21}^{-1} r_1 1 & & \\ & \ddots & \\ & & r_{2n}^{-1} r_{1n} \end{pmatrix}$$

Note that for clarity, we neglect to address the case of a component of r_2 being zero, in which case we would set the corresponding diagonal entry in A to zero, and the remainder of the proof holds. Therefore we have

$$r_1 \otimes c_1 + r_2 \otimes c_2 = r_2 A \otimes c_1 + r_2 \otimes c_2 = r_2 \otimes Ac_1 + r_2 \otimes c_2 = r_2 \otimes (Ac_1 + c_2)$$

□

Now let $r_1 \otimes c_1 + \cdots + r_n \otimes c_n$ in $R_n \otimes_{M_n(F)} C_n$ be such that $\phi(r_1 \otimes c_1 + \cdots + r_n \otimes c_n) = 0$. By Lemma 2.2 there are $r \in R_n$ and $c \in C_n$ such that $r_1 \otimes c_1 + \cdots + r_n \otimes c_n = r \otimes c$. By Lemma 2.1 $r_1 \otimes c_1 + \cdots + r_n \otimes c_n = 0$ since $\phi(r_1 \otimes c_1 + \cdots + r_n \otimes c_n) = \phi(r \otimes c) = 0$.

Hence ϕ is an injective linear transformation with a 1-dimensional vector space as a codomain. Then, since ϕ is not the zero map, e.g.

$$\phi((1, 0, \ldots, 0) \otimes (1, 0, \ldots, 0)^t) = 1 \neq 0$$

$R_n \otimes_{M_n(F)} C_n$ must also be a one dimensional vector space. Furthermore, in light of Lemma 2.2, ϕ is our explicit isomorphism we need, mapping $r \otimes c$ to rc.

3

Let F be a field and U, V, and W be vectors spaces over F. Furthermore let $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, and $w, w_1, w_2 \in W$.

(a)

Let $\beta : U \times V \times W \to U \otimes_F (V \otimes_F W)$ be the map defined by $\beta(u, v, w) = u \otimes (v \otimes w)$. By properties of the tensor product, we have

$$\begin{aligned} \beta(au + u', v, w) &= (au + u') \otimes (v \otimes w) \\ &= ((au) \otimes (v \otimes w)) + (u' \otimes (v \otimes w)) \\ &= a\,(u \otimes (v \otimes w)) + \beta(u', v, w) \\ &= a\beta(u, v, w) + \beta(u', v, w) \end{aligned}$$

$$
\begin{aligned}
\beta(u, av+v', w) &= u \otimes ((av+v') \otimes w) \\
&= u \otimes ((av \otimes w) + v' \otimes w) \\
&= u \otimes (a(v \otimes w) + v' \otimes w) \\
&= u \otimes (a(v \otimes w)) + u \otimes (v' \otimes w) \\
&= a(u \otimes (v \otimes w)) + \beta(u, v', w) \\
&= a\beta(u, v, w) + \beta(u, v', w)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta(u, av+v', w) &= u \otimes (v \otimes (aw + w')) \\
&= u \otimes ((v \otimes aw) + v \otimes w') \\
&= u \otimes (a(v \otimes w) + v \otimes w') \\
&= u \otimes (a(v \otimes w)) + u \otimes (v \otimes w')) \\
&= a(u \otimes (v \otimes w)) + \beta(u, v, w') \\
&= a\beta(u, v, w) + \beta(u, v, w')
\end{aligned}
$$

(b)

Let $T : U \times V \times W \to X$ be a F-trilinear map where X is some F-vector space. Then for a fixed $u \in U$, we can define $T_u : V \times W \to X$ by $T_u(v, w) = T(u, v, w)$. Since T is F-trilinear, then T_u is F-balanced. Thus, since X is an abelian group, the universal property of tensor products admits a *unique* group homomorphism $\varphi_u : V \otimes W \to X$ such that

$$T_u = \varphi_u \circ i \tag{3.1}$$

where $i : V \times W \to V \otimes W$ is the "inclusion" map $(v, w) \mapsto v \otimes w$; i.e. the diagram

$$
\begin{tikzcd}
V \times W \arrow[r, "i"] \arrow[dr, "T_u"'] & V \otimes W \arrow[d, "\varphi_u"] \\
& X
\end{tikzcd}
$$

commutes.

Next, we will use these φ_u maps to obtain the f for which we're looking. So define $\phi : U \times (V \otimes W) \to X$ by $\phi(u, v, w) = \varphi_u(v \otimes w)$. Then equation 3.1 gives us

$$
\begin{aligned}
\phi(u_1 + u_2, v \otimes w) &= \varphi_{u_1+u_2}(v \otimes w) \\
&= T_{u_1+u_2}(v, w) \\
&= T(u_1 + u_2, v, w) \\
&= T(u_1, v, w) + T(u_2, v, w) \\
&= T_{u_1}(v, w) + T_{u_2}(v, w) \\
&= \varphi_{u_1}(v \otimes w) + \varphi_{u_2}(v \otimes w) \\
&= \phi(u_1, v \otimes w) + \phi(u_2, v \otimes w)
\end{aligned}
$$

through the use of the F-trilinearity of T. We also have

$$
\begin{aligned}
\phi(au, v \otimes w) &= \varphi_{au}(v \otimes w) \\
&= T_{au}(v, w) \\
&= T(au, v, w) \\
&= T(u, av, w) \\
&= T_u(av, w)
\end{aligned}
$$

$$
\begin{aligned}
&= \varphi_u(av \otimes w) \\
&= \phi_u(a(v \otimes w))
\end{aligned}
$$

by the F-trilinearity of T, properties of $\otimes$, and equation 3.1. Finally, the homomorphic properties of each φ_u give us

$$
\begin{aligned}
\phi(u, v_1 \otimes w_1 + v_2 \otimes w_2) &= \varphi_u(v_1 \otimes w_1 + v_2 \otimes w_2) \\
&= \varphi_u(v_1 \otimes w_1) + \varphi_u(v_2 \otimes w_2) \\
&= \phi(u, v_1 \otimes w_1) + \phi(u, v_2 \otimes w_2)
\end{aligned}
$$

Given these three equations above, we conclude that ϕ is F-balanced. Hence the universal property of tensor products yields a *unique* group homomorphism $f : U \otimes (V \otimes W) \to X$ (i.e. a linear map) such that $\phi = f \circ i'$ is the inclusion map $(u, v \otimes w) \mapsto (u \otimes (v \otimes w))$. So by defining $\psi : U \times (V \times W) \to U \times (V \otimes W)$ as $\psi(u, v, w) = (u, v \otimes w)$, we have $\phi \circ \psi = f \circ i' \circ \psi$. However, since

$$\phi \circ e(u, v, w) = \phi(u, v \otimes w) = \varphi_u(v \otimes w) = T_u(v, w) = T(u, v, w)$$

and

$$i' \circ \psi(u, v, w) = i'(u, v \otimes w) = u \otimes (v \otimes w)$$

the equation $\phi \circ \psi = f \circ i' \circ \psi$ implies $T = f \circ \beta$.

(c)

Due to the symmetry of the situation, it is a laborious plug-and-chug operation to prove that

> Given a F-trilinear map $T : U \times V \times W \to X$ where X is some F-vector space, there exists a *unique* F-linear map $f : (U \otimes V) \otimes W \to X$ such that $T = f \circ \beta$

given that we already have the above proof in our hands. So we omit the proof and simply admit the above statement as fact. As there is already grounds for multi-linear universal property of tensor products [Lan02, p. 603], we will overload "the universal property of tensors products" by referring to the above statement, the statement proven in the previous part of this problem, and the original two-dimensional property as "the universal property of tensors products".

So by part (a) of this problem, we have that $\beta : U \times V \times W \to U \otimes (V \otimes W)$ is F-trilinear. The same argument holds, by shuffling around parentheses, for $\beta' : U \times V \times W \to (U \otimes V) \otimes W$, implying that it is F-trilinear. Therefore, the universal property of tensor products implies that there exist unique group homomorphisms $\alpha' : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ and $\alpha : U \otimes (V \otimes W) \to (U \otimes V) \otimes W$ such that $\alpha'((u \otimes v) \otimes w) = \beta(u, v, w)$ and $\alpha(u \otimes (v \otimes w)) = \beta'(u, v, w)$, i.e. the following diagrams commute

$$
\begin{tikzcd}
U \times V \times W \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & U \otimes (V \otimes W) \arrow[d, "\alpha"] \\
& (U \otimes V) \otimes W
\end{tikzcd}
\qquad
\begin{tikzcd}
U \times V \times W \arrow[r, "\beta'"] \arrow[dr, "\beta"'] & (U \otimes V) \otimes W \arrow[d, "\alpha'"] \\
& U \otimes (V \otimes W)
\end{tikzcd}
$$

But the uniqueness implies that α' and α are inverses of each. Thus α is the desired isomophism between $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$.

4

(a)

Let's start off by defining $\Phi_{\varphi\phi} : U \times V \to F$ for $\varphi \in U^\vee$ and $\phi \in V^\vee$ by $\Phi_{\varphi\phi}(u,v) = \varphi(u)\phi(v)$. This map is F-bilinear by the following three equations.

$$\begin{aligned}
\Phi_{\varphi\phi}(u_1+u_2,v) &= \varphi(u_1+u_2)\phi(v) \\
&= \varphi(u_1)\phi(v)+\varphi(u_2)\phi(v) \\
&= \Phi_{\varphi\phi}(u_1,v)+\Phi_{\varphi\phi}(u_2,v) \\
\Phi_{\varphi\phi}(u,v_1+v_2) &= \varphi(u)\phi(v_1+v_2) \\
&= \varphi(u)\phi(v_1)+\varphi(u)\phi(v_2) \\
&= \Phi_{\varphi\phi}(u,v_1)+\Phi_{\varphi\phi}(u,v_2) \\
\Phi_{\varphi\phi}(au,v) &= \varphi(au)\phi(v) \\
&= r\varphi(u)\phi(v) \\
&= \varphi(u)\phi(av) \\
&= \Phi_{\varphi\phi}(u,av)
\end{aligned}$$

Thus the universal property of tensor products gives us a unique linear map $\overline{\Phi_{\varphi\phi}} : U \otimes V \to F$ through which Φ factors. We use this in the following argument.

Now define $f : U^\vee \times V^\vee \to (U \otimes V)^\vee$ by $f(\varphi,\phi) = \overline{\Phi}_{\varphi\phi}$. We yet again have an F-bilinear map here by the following equations, using what we have shown above.

$$\begin{aligned}
f(\varphi_1+\varphi_2,\phi)(u,v) &= \Phi_{(\varphi_1+\varphi_2)\phi}(u,v) \\
&= \varphi_1(u)\phi(v)+\varphi_2(u)\phi(v) \\
&= \Phi_{\varphi_1\phi}(u,v)+\Phi_{\varphi_2\phi}(u,v) \\
&= f(\varphi_1,\phi)(u,v)+f(\varphi_2,\phi)(u,v) \\
f(\varphi,\phi_1+\phi_2)(u,v) &= \Phi_{\varphi(\phi_1+\phi_2)}(u,v) \\
&= \varphi(u)\phi_1(v)+\varphi(u)\phi_2(v) \\
&= \Phi_{\varphi\phi_1}(u,v)+\Phi_{\varphi\phi_2}(u,v) \\
&= f(\varphi,\phi_1)(u,v)+f(\varphi,\phi_2)(u,v) \\
f(a\varphi,\phi)(u,v) &= \Phi_{(a\varphi)\phi}(u,v) \\
&= (a\varphi)(u)\phi(v) \\
&= a(\varphi(u)\phi(v)) \\
&= \varphi(u)(a\phi)(v) \\
&= \Phi_{\varphi(a\phi)}(u,v) \\
&= f(\varphi,a\phi)(u,v)
\end{aligned}$$

Thus the universal property of tensor products gives us a unique linear map $\overline{f} : U^\vee \otimes V^\vee \to (U \otimes V)^\vee$ through which f factors.

Now we define $g : (U \otimes V)^\vee \to U^\vee \otimes V^\vee$ (this will be our inverse of $\overline{f}$) to be the map

$$\overline{\alpha} \mapsto (u \mapsto \alpha(u,1)) \otimes (v \mapsto \alpha(1,v))$$

where $\alpha : U \times V \to F$ is the map associated with $\overline{\alpha}$ according to the universal property of tensor products. By the following we see that g is the inverse of $\overline{f}$

$$\begin{aligned}
g(\overline{f}(\varphi\otimes\phi)) &= g(\Phi_{\varphi\phi}) \\
&= (u \mapsto \Phi_{\varphi\phi}(u,1)) \otimes (v \mapsto \Phi_{\varphi\phi}(1,v)) \\
&= (u \mapsto \varphi(u)) \otimes (v \mapsto \phi(v))
\end{aligned}$$

$$= \quad \varphi \otimes \phi$$

and so we have that $\overline{f}$ is the isomorphism which we desire.

(b) Extra Credit

References

[DF04] D.S. Dummit and R.M. Foote. *Abstract Algebra*. John Wiley & Sons Canada, Limited, 2004.

[Lan02] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2002.