Math 503: Abstract Algebra
Homework 3

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1

Define $D(F) \in \mathbb{Z}[x_1, \ldots, x_n]$ by

$$D(F) = \sum_{1 \le i < j \le n} (x_i - x_j)^2 \tag{1.1}$$

(a)

Existence Given equation 1.1, $D(F)$ can alternatively be specified as

$$D(F) = \sum_{i,j \in \{1,\ldots,n\},\ i<j} (x_i - x_j)^2$$

Because any $\sigma \in S_n$ is bijective and because $(x_i - x_j)^2 = ((-1)(x_j - x_i))^2 = (-1)^2(x_j - x_i)^2 = (x_j - x_i)^2$, then

$$\sum_{i,j \in \{1,\ldots,n\},\ i<j} (x_{\sigma(i)} - x_{\sigma(j)})^2 = \sum_{i,j \in \{1,\ldots,n\},\ i<j} (x_i - x_j)^2 = D(F)$$

In other words, $D(F)$ is symmetric. Thus there indeed exists a polynomial $d \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $d(s_1, \ldots, s_n) = D(F)$ since every symmetric polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ is contained in $\mathbb{Z}[s_1, \ldots, s_n]$.

Uniqueness There cannot exist two distinct $d_1, d_2 \in \mathbb{Z}[x_1, \ldots, x_n]$ such that $d_1(s_1, \ldots, s_n) = d_2(s_1, \ldots, s_n) = D(F)$ because their existence would contradict the algebraic independence of $s_1, \ldots, s_n$ [Lan02, p. 192] since it would imply $d = d_1 - d_2$ has $d(s_1, \ldots, s_n) = 0$.

(b)

Degree $n = 2$ To find an explicit formula for disc(f) where $f(t)$ is a monic polynomial of degree two, we need to find $d(z_1, z_2) \in \mathbb{Z}[z_1, z_2]$ such that $d(s_1, s_2) = (x_1 - x_2)^2$. Because of the existence/uniqueness proven above, we can find such a polynomial and there will be only one. So because $s_1 = x_1 + x_2$, $s_2 = x_1 x_2$ for $n = 2$ and because

$$(x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2 = (x_1^2 + 2x_1x_2 + x_2^2) - 4x_1x_2 = (x_1 + x_2)^2 - 4x_1x_2$$

then we can deduce that $d(z_1, z_2) = z_1^2 - 4z_2$. Hence for any $f(t) = t^2 + at + b$

$$\mathrm{disc}(f) = d(-a, b) = a^2 - 4b$$

Degree $n = 3$ To find an explicit formula for disc(f) where $f(t)$ is a monic polynomial of degree three, we need to find $d(z_1, z_2, z_3) \in \mathbb{Z}[z_1, z_2, z_3]$ such that $d(s_1, s_2, s_3) = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$. Again, because of the existence/uniqueness proven above, we can find such a polynomial and there will be only one.

We first notice that $d(s_1, s_2, s_3)$ will be a homogeneous polynomial of degree six in x_1, x_2, x_3. Therefore, since $s_1 = x_1 + x_2 + x_3$, $s_2 = x_1x_2 + x_1x_3 + x_2x_3$, and $s_3 = x_1x_2x_3$,

$$d(s_1, s_2, s_3) = \alpha_1 s_3^2 + \alpha_2 s_3 s_2 s_1 + \alpha_3 s_3 s_1^3 + \alpha_4 s_2^3 + \alpha_5 s_2^2 s_1^2 + \alpha_6 s_2 s_1^4 + \alpha_7 s_1^6$$

for some integers $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$. We will find these values by analyzing the following expansion

$$\begin{aligned}(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = {} & x_1^4x_2^2 - 2x_1^3x_2^3 + x_1^2x_2^4 - 2x_1^4x_2x_3 + 2x_1^3x_2^2x_3 \\ & + 2x_1^2x_2^3x_3 - 2x_1x_2^4x_3 + x_1^4x_3^2 + 2x_1^3x_2x_3^2 - 6x_1^2x_2^2x_3^2 \\ & + 2x_1x_2^3x_3^2 + x_2^4x_3^2 - 2x_1^3x_3^3 + 2x_1^2x_2x_3^3 + 2x_1x_2^2x_3^3 \\ & - 2x_2^3x_3^3 + x_1^2x_3^4 - 2x_1x_2x_3^4 + x_2^2x_3^4\end{aligned} \tag{1.2}$$

From this equation, it is immediately apparent that $\alpha_6 = \alpha_7 = 0$ since $s_2 s_1^4$ and s_1^6 would produce monomials containing x_1 raised to a degree higher than 4, but there is no such monomial in the polynomial of equation 1.2. Now, given the expansions

$$\begin{aligned}
s_3^2 &= x_1^2x_2^2x_3^2 \\
s_3s_2s_1 &= x_1^3x_2^2x_3 + x_1^2x_2^3x_3 + x_1^3x_2x_3^2 + 3x_1^2x_2^2x_3^2 + x_1x_2^3x_3^2 + x_1^2x_2x_3^3 + x_1x_2^2x_3^3 \\
s_3s_1^3 &= x_1^4x_2x_3 + 3x_1^3x_2^2x_3 + 3x_1^2x_2^3x_3 + x_1x_2^4x_3 + 3x_1^3x_2x_3^2 + 6x_1^2x_2^2x_3^2 + 3x_1x_2^3x_3^2 + 3x_1^2x_2x_3^3 \\
&\quad +3x_1x_2^2x_3^3 + x_1x_2x_3^4 \\
s_2^3 &= x_1^3x_2^3 + 3x_1^3x_2^2x_3 + 3x_1^2x_2^3x_3 + 3x_1^3x_2x_3^2 + 6x_1^2x_2^2x_3^2 + 3x_1x_2^3x_3^2 + x_1^3x_3^3 + 3x_1^2x_2x_3^3 \\
&\quad +3x_1x_2^2x_3^3 + x_2^3x_3^3 \\
s_2^2s_1^2 &= x_1^4x_2^2 + 2x_1^3x_2^3 + x_1^2x_2^4 + 2x_1^4x_2x_3 + 8x_1^3x_2^2x_3 + 8x_1^2x_2^3x_3 + 2x_1x_2^4x_3 + x_1^4x_3^2 + 8x_1^3x_2x_3^2 \\
&\quad +15x_1^2x_2^2x_3^2 + 8x_1x_2^3x_3^2 + x_2^4x_3^2 + 2x_1^3x_3^3 + 8x_1^2x_2x_3^3 + 8x_1x_2^2x_3^3 + 2x_2^3x_3^3 + x_1^2x_3^4 \\
&\quad +2x_1x_2x_3^4 + x_2^2x_3^4
\end{aligned}$$

we will strategically evaluate the coefficients of certain monomials in order to give us five linear equations that will allow us to solve for the α variables. So for example, the coefficient in equation 1.2 of $x_1^2x_2^2x_3^2$ is -6, and in the expansions of s_3^2, $s_3s_2s_1$, $s_3s_1^3$, s_2^3, and $s_2^2s_1^2$ the coefficients are 1, 3, 6, 6, and 15, respectively. Hence we have the equation

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 + 6\alpha_4 + 15\alpha_5 = -6$$

In the same vein, we evaluate the coefficients for the monomials $x_1^3x_2^2x_3$ to get

$$\alpha_2 + 3\alpha_3 + 3\alpha_4 + 8\alpha_5 = 2$$

$x_1^4x_2x_3$ to get

$$\alpha_3 + 2\alpha_5 = -2$$

$x_1^3x_3^3$ to get

$$\alpha_4 + 2\alpha_5 = -2$$

and finally $x_1^4x_3^2$ to get

$$\alpha_5 = 1$$

Solving the above five equations we obtain $\alpha_1 = -27$, $\alpha_2 = 18$, $\alpha_3 = -4$, $\alpha_4 = -4$, and $\alpha_5 = 1$, which in turn informs us that

$$d(z_1, z_2, z_3) = -27z_3^2 + 18z_3z_2z_1 - 4z_3z_1^3 - 4z_2^3 + z_1^2z_2^2$$

Hence for any $f(t) = t^3 + at^2 + bt + c$

$$\text{disc}(f) = d(-a, b, -c) = -27c^2 + 18abc - 4a^3c - 4b^3 + a^2b^2$$

(c) Extra Credit

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2 Newton Polynomials

(a) Find formulas for Newton polynomials p_2, p_3, and p_4 in terms of s_1 through s_4

Since p_2, p_3, and p_4 are symmetric polynomials over $\mathbb{Z}$, they can be uniquely represented as a polynomial in terms of the elementary symmetric polynomials over $\mathbb{Z}$. Furthermore, they are homogeneous polynomials of degrees 2, 3, and 4, respectively, and must therefore have the forms

$$\begin{aligned}
p_2 &= a_1s_1^2 + a_2s_2 \\
p_3 &= b_1s_1^3 + b_2s_1s_2 + b_3s_3 \\
p_4 &= c_1s_1^4 + c_2s_1^2s_2 + c_3s_2^2 + c_4s_4
\end{aligned}$$

Clause	First Term, Lexicographically
s_1^2	x_1^2
s_2	x_1x_2
s_1^3	x_1^3
s_1s_2	$x_1^2x_2$
s_3	$x_1x_2x_3$
s_1^4	x_1^4
$s_1^2s_2$	$x_1^3x_2$
s_2^2	$x_1^2x_2^2$
s_4	$x_1x_2x_3x_4$

Table 2.1: First terms of elementary symmetric polynomials that generate Newton polynomials.

for integers a_i, b_i, and c_i.

It's clear that in order to obtain the appropriate values of p_2, p_3, and p_4, $a_1 = b_1 = c_1 = 1$ since none of the other clauses in the above equations will be able to account for the x_i^2, x_i^3, and x_i^4 in each of p_1, p_2, and p_3. Hence, in determining the formula for p_i, our approach will be to start with s_1^i ordered lexicographically and find the coefficient of the leftmost clause that isn't in p_i and then subtract the product of that coefficient with the appropriate clause on the right hand side of the equations above. We will know which are appropriate by looking at the lexicographical first term of each of the clauses: where these terms have been generated for $n = 4$.
So for the case of p_2 we start with

$$s_1^2 = x_1^2 + 2x_1x_2 + x_2^2 + 2x_1x_3 + 2x_2x_3 + x_3^2 + 2x_1x_4 + 2x_2x_4 + 2x_3x_4 + x_4^2$$

in which the first term x_1^2 should remain since it is in p_2, but we need to get rid of the second term $2x_1x_2$. Through use of Table 2.1, this implies that $a_2 = -2$ so that we are left with $s_1^2 - 2s_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and thus the formula for p_2.

For the case of p_3 we start with $s_1^3 = x_1^4 + 3x_1^2x_2 + \cdots$, leaving off the, currently unimportant, terms after the first two. Again through use of Table 2.1, the coefficient of 3 in the second term informs us that $b_2 = -3$ so that we are left with $s_1^3 - 3s_2s_1 = x_1^3 + x_2^3 - 3x_1x_2x_3 + \cdots$, which finally informs us that $b_3 = 3$, leaving us with p_3's formula: $s_1^3 - 3s_2s_1 + 3s_2$.

Finally, by following the same procedure regarding p_4, we have

$$\begin{aligned} s_1^4 &= x_1^4 + 4x_1^3x_2 + \cdots \\ s_1^4 - 4s_1^2s_2 &= x_1^4 - 2x_1^2x_2^2 + \cdots \\ s_1^4 - 4s_1^2s_2 + 2s_2^2 &= x_1^4 + x_2^4 - 4x_1^2x_2x_3 + \cdots \end{aligned}$$

which therefore implies $s_1^4 - 4s_1^2s_2 + 2s_2^2 + 4s_4 = x_1^4 + x_2^4 + x_3^4 + x_4^4$ and yields the formula for p_4.

(b) Extra Credit

(c) Extra Credit

3

Let R be a commutative ring and M be an R-module.

(a) Show the existence of an R-linear "permutation action" of S_n on $\otimes_R^n M$

For each $\sigma \in S_n$, define $f_\sigma : M^n \to M^n$ by $(x_1, \ldots, x_n) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for $x_1, \ldots, x_n \in M$. With this definition

$$\begin{aligned} f_\sigma((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) &= f_\sigma(x_1 + y_1, \ldots, x_n + y_n) \\ &= f_\sigma(z_1, \ldots, z_n) \\ &= \left(z_{\sigma(1)}, \ldots, z_{\sigma(n)}\right) \\ &= \left(x_{\sigma(1)} + y_{\sigma(1)}, \ldots, x_{\sigma(n)} + y_{\sigma(n)}\right) \\ &= \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) + \left(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right) \\ &= f_\sigma\left(x_1, \ldots, x_n\right) + f_\sigma\left(y_1, \ldots, y_n\right) \end{aligned}$$

for $x_1, \ldots, x_n, y_1, \ldots, y_n \in M$ letting $z_i = x_i + y_i$ for each such i. Furthermore,

$$\begin{aligned} f_\sigma(r(x_1, \ldots, x_n)) &= f_\sigma(rx_1, \ldots, rx_n) \\ &= f_\sigma(y_1, \ldots, y_n) \\ &= \left(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right) \\ &= \left(rx_{\sigma(1)}, \ldots, rx_{\sigma(n)}\right) \\ &= r\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right) \\ &= rf_\sigma\left(x_1, \ldots, x_n\right) \end{aligned}$$

for $r \in R$ and $x_1, \ldots, x_n \in M$ letting $y_i = rx_i$ for each such i. Therefore each f_σ is an R-multilinear map yielding, via the universal property of tensors, the existence of a unique R-linear homomorphism $\overline{f}_\sigma : \otimes_R^n M \to M^n$ through which f_σ factors. Hence $\overline{f}_\sigma(x_1 \otimes \cdots \otimes x_n) = \left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right)$ for each $\sigma \in S_n$ and therefore by defining the action of S_n on $\otimes_R^n M$ by having σ act on $\mathbf{x} \in \otimes_R^n M$ by $(i \circ \overline{f}_\sigma)(\mathbf{x})$ where $i : M^n \to \otimes_R^n M$ is the normal R-multilinear inclusion map, we obtain the desired R-linear permutation action, since $i \circ \overline{f}_\sigma$ satisfies the axioms of a group action, being that i and $\overline{f}_\sigma$ are each homomorphisms. Note that we omit the use of $\sigma \cdot \mathbf{x}$ or $\mathbf{x} \cdot \sigma$ notation in light of the next part of this problem.

(b) Is the action previously defined a left or right action?

Set $\varphi_\sigma = i \circ \overline{f}_\sigma$ for each $\sigma \in S_n$ so that $\varphi_\sigma\left(x_1 \otimes \cdots \otimes x_n\right) = i \circ \overline{f}_\sigma\left(x_1 \otimes \cdots \otimes x_n\right) = \left(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}\right)$ for each $x_1, \ldots, x_n \in M$. For repetitious use later, we point out that $\varphi_{\tau\sigma} = \varphi_\tau \circ \varphi_\sigma$ for all $\tau, \sigma \in S_n$ according to

$$\begin{aligned} \varphi_{\tau\sigma}(\mathbf{x}) &= \varphi_{\tau\sigma}\left(\sum_i \mathbf{x}_i\right) \\ &= \sum_i \varphi_{\tau\sigma}\left(\mathbf{x}_i\right) \\ &= \sum_i x_{i\tau\sigma(1)} \otimes \cdots \otimes x_{i\tau\sigma(n)} \\ &= \sum_i x_{i\tau(\sigma(1))} \otimes \cdots \otimes x_{i\tau(\sigma(n))} \\ &= \sum_i \varphi_\tau\left(x_{i\sigma(1)} \otimes \cdots \otimes x_{i\sigma(n)}\right) \\ &= \sum_i \varphi_\tau \circ \varphi_\sigma\left(\mathbf{x}_i\right) \\ &= \varphi_\tau \circ \varphi_\sigma \sum_i \mathbf{x}_i \\ &= \varphi_\tau \circ \varphi_\sigma(\mathbf{x}) \end{aligned} \tag{3.3}$$

for each $\mathbf{x} = \sum_i \mathbf{x}_i = \sum_i x_{i1} \otimes \cdots \otimes x_{in} \in \otimes_R^n M$. With equation 3.3 in hand we see that if we were to attempt to make this action a left action then

$$\tau\sigma \cdot \mathbf{x} = \varphi_{\tau\sigma}\mathbf{x} = \varphi_\tau\varphi_\sigma(\mathbf{x}) = \varphi_\tau\left(\varphi_\sigma(\mathbf{x})\right) = \tau \cdot \left(\varphi_\sigma(\mathbf{x})\right) = \tau \cdot (\sigma \cdot \mathbf{x})$$

and if we were to do so as a right action then

$$\mathbf{x}\cdot\tau\sigma = \varphi_{\tau\sigma}\mathbf{x} = \varphi_\tau\varphi_\sigma(\mathbf{x}) = \varphi_\tau\left(\varphi_\sigma(\mathbf{x})\right) = \left(\varphi_\sigma(\mathbf{x})\right)\cdot\tau = (\mathbf{x}\cdot\sigma)\cdot\tau$$

for each $\mathbf{x}\in\otimes_R^n M$, implying not only that this action is a left action, but that it is *not* are right action.

(c)

(d) Show $R[y_1,\dots,y_n]\cong S_R^\bullet(M)$ for free R-module M with $\operatorname{rank}(M)=n$

Since both $R[y_1,\dots,y_n]$ is a graded ring where each homogenous component of degree k is the ring of homogenous polynomials of degree k (denote it $R^k[y_1,\dots,y_n]$) and $S_R^\bullet(M)$ is a graded ring where the homogeneous components of degree k is $S^k(M)$, it suffices to prove that $R^k[y_1,\dots,y_n]$ is isomorphic to $S^k(M)$ for arbitrary k.

Let $\mathcal{B}=\{v_1,\dots,v_n\}$ be a set of free generators on M and $Y=\{y_1,\dots,y_n\}$. We will use $\bullet$ to denote the tensor operation in $S^k(M)$. Define $\alpha: M^k\to R^k[y_1,\dots,y_n]$ and $\beta: Y\to S^k(M)$ by

$$\begin{aligned}\alpha(m_1,\dots,m_k) &= \prod_i\left(\sum_{j=1}^n \delta_j(m_i)x_j\right)\\ \beta(x_i) &= v_i\bullet 1\bullet\cdots\bullet 1\end{aligned}$$

where $\delta_i\in\operatorname{Hom}_{R-\mathrm{mod}}(M,R)$ is the linear operator which takes $v_i\in\mathcal{B}$ to 1 and all other elements of $\mathcal{B}$ to zero. With the product in its definition and because each factor in that product is the sum of linear maps, α is a symmetric k-multilinear map. Therefore, we have that two results:

1. The existence of an R-module homomorphism $\overline{\alpha}: S^k(M)\to R^k[y_1,\dots,y_n]$ through which α factors. This comes by way of the universal property of symmetric multilinear maps.
2. The existence of an R-algebra homomorphism $\overline{\beta}: R^k[y_1,\dots,y_n]\to S^k(M)$ through which β factors. This is given by the universal property of polynomial algebras.

Pictorially we have the following commutative diagram.

$$\begin{tikzcd} M^k \arrow[rr] \arrow[drr, "\alpha"'] & & S^k(M) \arrow[d, shift right, "\overline{\alpha}"'] & \\ & & R^k[y_1,\cdots,y_n] \arrow[u, shift right, "\overline{\beta}"'] & & Y \arrow[ll] \arrow[ull, "\beta"'] \end{tikzcd}$$

Now, let's denote $\underbrace{x\bullet\cdots\bullet x}_{j\text{ times}}\in S^k(M)$ by $x^{\bullet j}$ so that $x^{\bullet j}=x^j\bullet\underbrace{1\bullet\cdots\bullet 1}_{j-1\text{ times}}$ since each element of $S^k(M)$ is symmetric. Thus for the generators of $S^k(M)$ and $R^k[y_1,\cdots,y_n]$

$$\overline{\alpha}\left(\overline{\beta}\left(y_1^{a_1}\cdots y_n^{a_n}\right)\right) = \overline{\alpha}\left(\overline{\beta}(y_1)^{\bullet a_1}\cdots\overline{\beta}(y_n)^{\bullet a_n}\right) = \overline{\alpha}\left(v_1^{\bullet a_1}\cdots v_n^{\bullet a_n}\right) = y_1^{a_1}\cdots y_n^{a_n}$$

and

$$\overline{\beta}\left(\overline{\alpha}\left(v_{i_1}\bullet\cdots\bullet v_{i_k}\right)\right) = \overline{\beta}\left(\alpha\left(v_{i_1},\dots,v_{i_k}\right)\right) = \overline{\beta}\left(x_{i_1}\cdots x_{i_k}\right) = v_{i_1}\bullet\cdots\bullet v_{i_k}$$

which implies that $\overline{\alpha}$ and $\overline{\beta}$ are inverses of each other. Hence $S^k(M)$ is isomorphic to $R^k[y_1,\cdots,y_n]$.

(e) Extra Credit

4

(a) Compute the character of $\boldsymbol{\rho_k}$

Denote the symmetric and rotational generators of D_{2n} by s and r, respectively. Since ρ is the homomorphism of the action of D_{2n} acting on $V_1 = \mathbb{R}^2$, then for any arbitrary element $s^i r^j \in D_{2n}$, $i \in \{0, 1\}$, $j \in \{0, \dots, n-1\}$

$$\rho(s^i r^j) = \rho(s)\rho(r^j) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}^i \begin{pmatrix} \cos\left(\frac{2\pi j}{n}\right) & -\sin\left(\frac{2\pi j}{n}\right) \\ \sin\left(\frac{2\pi j}{n}\right) & \cos\left(\frac{2\pi j}{n}\right) \end{pmatrix}$$

and therefore

$$\chi_\rho(s^i r^j) = \operatorname{Tr}\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}^i \begin{pmatrix} \cos\left(\frac{2\pi j}{n}\right) & -\sin\left(\frac{2\pi j}{n}\right) \\ \sin\left(\frac{2\pi j}{n}\right) & \cos\left(\frac{2\pi j}{n}\right) \end{pmatrix}\right) = \begin{cases} 0 & i = 1 \\ \cos\left(\frac{2\pi j}{n}\right) & \text{otherwise} \end{cases}$$

Hence we can compute the character of ρ_k of any $s^i r^j \in D_{2n}$

$$\begin{aligned} \chi_{\rho_k}(s^i r^j) &= \operatorname{Tr}\left(\rho_k(s^i r^j)\right) \cdots \chi\left(\rho(s^i r^j)\right) \\ &= \operatorname{Tr}\left(\rho(s^i r^j) \otimes \cdots \otimes \rho(s^i r^j)\right) \\ &= \operatorname{Tr}\left(\rho(s^i r^j)\right) \cdots \operatorname{Tr}\left(\rho(s^i r^j)\right) \\ &= \begin{cases} 0 & i = 1 \\ \cos^k\left(\frac{2\pi j}{n}\right) & \text{otherwise} \end{cases} \end{aligned}$$

(b)

(c)

(d) Extra Credit

References

[Lan02] S. Lang. *Algebra*. Graduate Texts in Mathematics. Springer New York, 2002.