Math 503: Abstract Algebra
Homework 4

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February 22, 2014
http://coursework.tylerlogic.com/courses/upenn/math503/homework04

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Let G be a finite group and S be a finite set on which G acts, and the action is given by $\mu : G \times S \to S$. Let V be the space of $\mathbb{C}$ valued functions on S. Let $\rho : G \to GL(V)$ be the linear representation of G defined by

$$(\rho_g(f))\,(s) = f(\mu(g^{-1}, s)) \qquad \forall g \in G, s \in S, f \in \mathbb{C}^S$$

Let χ_ρ be the character of (V, ρ).

Let the elements of S be $\{s_1, \cdots, s_m\}$, and denote by $\mathcal{B} = \{f_{s_i}\} \subset V$ the basis of functions such that $f_{s_i}(s_j) = \delta_{ij}$ for each $s_j \in S$.

(a) Prove that $(\chi_\rho | 1_G)$ is the number of conjugacey class in G

Fix an $s \in S$, and define f to be the vector of V by

$$f = \sum_{s_i \in Gs} f_{s_i}$$

Then for any $g \in G$ and $s_j \in S$

$$\rho_g(f)(s_j) = \sum_{s_i \in Gs} \rho_g\left(f_{s_i}\right) = \sum_{s_i \in Gs} f_{s_i}\left(\mu(g^{-1}, s_j)\right) = \begin{cases} 1 & \mu(g^{-1}, s_j) \in Gs \\ 0 & \text{otherwise} \end{cases}$$

However, if $g^{-1}s_j \in Gs$, then there exists a $g' \in G$ with $g^{-1}s_j = g's$, which implies $s_j = gg's$, meaning that s_j is also in Gs. Hence the above equation boils down to

$$\rho_g(f)(s_j) = \begin{cases} 1 & s_j \in Gs \\ 0 & \text{otherwise} \end{cases} = \sum_{s_i \in Gs} f_{s_i}(s_j) = f(s_j)$$

Thus the 1-dimensional subspace $W_s = \text{span}\{f\}$ is G-stable. Since there is one such W_s corresponding to each conjugacey class Gs, then $(\chi_\rho | \mathbf{1}_G)$ is the number of conjugacey classes. [1]

(b)

Now let G operate transitively on S, and define $U \subset V$ be the set of functions f for which $\sum_{s \in S} f(s) = 0$. Then for any $f \in U$, $f = a_1 f_{s_1} + \cdots + a_m f_{s_m}$

$$\sum_{s \in S} f(s) = \sum_{s \in S} a_1 f_{s_1}(s) + \cdots + a_m f_{s_m}(s) = a_1 + \cdots + a_m = 0$$

implying that $a_1 + \cdots + a_{m-1} = -a_m$, which further implies that U has dimension $\dim V - 1$.

Now because G is transitive, then for any $g \in G$

$$\sum_{s \in S} \rho_g(f)(s) = \sum_{s \in S} f(\mu(g^{-1}, s)) = \sum_{s \in S} f(s) = 0$$

implying that $\rho_g(f) \in U$, i.e. U is G-stable. Therefore V can be decomposed into $V = U \oplus W$ for some G-stable subspace W, by Maschke's theorem. However, since U has dimension $\dim V - 1$, then W has dimension 1, and since G is acts transitively on S, then $(\chi_\rho | \mathbf{1}_G) = 1$, implying that W must be the single trivial representation in the decomposition of V. Thus, denoting the character of U by χ_U,

$$\chi_U = \chi_\rho - 1$$

(c)

[1] Inspiration drawn from [Sho]

(d)

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(a)

(b)

(c)

(d) Extra Credit

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(a)

(b)

(c)

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(a)

(b) Extra Credit

References

[Sho] Clayton Shonkwiler. Algebra hw 2. http://www.math.uga.edu/~clayton/courses/503/503_3.pdf.