Math 503: Abstract Algebra
Homework 5

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February 27, 2014
http://coursework.tylerlogic.com/courses/upenn/math503/homework05

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(a) Compute the character table of the Heisenberg group $H(\mathbb{F}_3)$

Let $a, b, c, x, y, z \in \mathbb{F}_3$. Then

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & xz-y \\ & 1 & -z \\ & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & a & cx-az+b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

by which we see that

$$\left\{ \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\} \bigcup \left\{ \begin{pmatrix} 1 & a & \\ & 1 & c \\ & & 1 \end{pmatrix} \middle| a, c \in \mathbb{F}_3 \right\}$$

represent the conjugacy classes of $H(\mathbb{F}_3)$. Thus there are 11 irreducible characters. We will denote the conjugacy classes by C_0, C_1, C_2, C_{01}, C_{02}, C_{10}, C_{11}, C_{12}, C_{20}, C_{21}, and C_{22} where

$$C_b = \left\{ \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} \right\} \qquad \text{and} \qquad C_{ac} = \left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \middle| b \in \mathbb{F}_3 \right\}$$

We will denote the eleven characters by χ_1 through χ_{11}, with χ_1 being the trivial character. With this notation, we have the following steps of reasoning about the values of the character table.

1. The trivial character takes all elements to the identity. So we have:

size:	1	1	1	3	3	3	3	3	3	3	3
class:	C_0	C_1	C_2	C_{01}	C_{02}	C_{10}	C_{11}	C_{12}	C_{20}	C_{21}	C_{22}
χ_1	1	1	1	1	1	1	1	1	1	1	1

2. Since the degrees of the each character must divide the order of the group (which is 27) and the sum of the squares of the degrees must equal the order of the group, then the degrees of the irreducible characters have to be 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3 as $9(1^2) + 2(3^2) = 27$. This gives us the entire first column of the table.

3. Define the maps $\varphi : H(\mathbb{F}_3) \to \mathbb{F}_3$ and $\phi : H(\mathbb{F}_3) \to \mathbb{F}_3$ by

$$\varphi \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} = a \qquad \text{and} \qquad \phi \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} = c$$

Therefore $\ker \varphi = \left(\bigcup_{b \in \mathbb{F}_3} C_b\right) \cup \left(\bigcup_{c \in \mathbb{F}_3} C_{0c}\right)$ and $\ker \phi = \left(\bigcup_{b \in \mathbb{F}_3} C_b\right) \cup \left(\bigcup_{a \in \mathbb{F}_3} C_{a0}\right)$, which implies $H(\mathbb{F}_3)/\ker\varphi \cong \mathbb{F}_3$ and $H(\mathbb{F}_3)/\ker\phi \cong \mathbb{F}_3$. This, in turn, demands that the non-trivial degree-1 characters can be determined from the non-trivial degree-1 characters on $\mathbb{F}_3$. Define χ_2 to be the character corresponding to φ and the character on $\mathbb{F}_3$ mapping $i \mapsto \zeta^i$ where $\zeta = e^{\frac{2\pi i}{3}}$ is the third root of unity. Similarly let χ_3 be the character corresponding to ϕ and the character on $\mathbb{F}_3$ mapping $i \mapsto \zeta^i$. We thus have:

size:	1	1	1	3	3	3	3	3	3	3	3
class:	C_0	C_1	C_2	C_{01}	C_{02}	C_{10}	C_{11}	C_{12}	C_{20}	C_{21}	C_{22}
χ_2	1	1	1	1	1	ζ	ζ	ζ	ζ^2	ζ^2	ζ^2
χ_3	1	1	1	ζ	ζ^2	1	ζ	ζ^2	1	ζ	ζ^2

4. Given that χ_2 and χ_3 are non-trivial characters of dimension one, their tensor yields another non-trivial character of dimension one. We can then again tensor the resulting one dimensional character with other one-dimensional characters to get more one dimensional characters, and then repeat. In this vein we define:

$$\begin{aligned}
\chi_4 &= \chi_2\chi_3 \\
\chi_5 &= \chi_3\chi_4 \\
\chi_6 &= \chi_2\chi_4 \\
\chi_7 &= \chi_3\chi_6 \\
\chi_8 &= \chi_2\chi_7 \\
\chi_9 &= \chi_3\chi_7
\end{aligned}$$

and therefore we are only left with finding the values of χ_{10} and χ_{11}. This gives us:

size:	1	1	1	3	3	3	3	3	3	3	3
class:	C_0	C_1	C_2	C_{01}	C_{02}	C_{10}	C_{11}	C_{12}	C_{20}	C_{21}	C_{22}
χ_4	1	1	1	ζ	ζ^2	ζ	ζ^2	1	ζ^2	1	ζ
χ_5	1	1	1	ζ^2	ζ	ζ	1	ζ^2	ζ^2	ζ	1
χ_6	1	1	1	ζ	ζ^2	ζ^2	1	ζ	ζ	ζ^2	1
χ_7	1	1	1	ζ^2	ζ	ζ^2	ζ	1	ζ	1	ζ^2
χ_8	1	1	1	ζ^2	ζ	1	ζ^2	ζ	1	ζ^2	ζ
χ_9	1	1	1	1	1	ζ^2	ζ^2	ζ^2	ζ	ζ	ζ

5. For each conjugacy class which is not one of C_0, C_1, or C_2, the current values of χ_1 through χ_9, when paired with themselves and multiplied by the size of the conjugacy class (all have size 3), gives the size of the group. Therefore the values of χ_{10} and χ_{11} must all be zero on these conjugacy classes. We are now left with finding the values of χ_{10} and χ_{11} on C_1 and C_2.

   Now because
$$\begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & 2 \\ & 1 & \\ & & 1 \end{pmatrix}$$
   and $\overline{\chi(x)} = \chi(x^{-1})$ for any $x \in G$, then both $\chi_{10}(x_1) = \overline{\chi_{10}(x_2)}$ and $\chi_{11}(x_1) = \overline{\chi_{11}(x_2)}$ for all $x_1 \in C_1$, $x_2 \in C_2$. Furthermore, the orthogonality relations tell us that for $x_1 \in C_1$ and $x_2 \in C_2$ (denoting $\chi_{10}(x_1)$, $\chi_{11}(x_2)$ by α, β respectively)

   (a) Pairing the first and second column yields $\alpha + \beta = -3$

   (b) Pairing the first and third column yields $\overline{\alpha} + \overline{\beta} = -3$

   (c) Pairing the second to last row with itself yields $\alpha\overline{\alpha} = 9$, i.e. $|\alpha| = 3$

   (d) Pairing the last row with itself yields $\beta\overline{\beta} = 9$, i.e. $|\beta| = 3$

   By the above constraints, we can see that $\chi_{10}(x_1) = \chi_{11}(x_2) = 3\zeta$ is a viable solution. This yields

size:	1	1	1	3	3	3	3	3	3	3	3
class:	C_0	C_1	C_2	C_{01}	C_{02}	C_{10}	C_{11}	C_{12}	C_{20}	C_{21}	C_{22}
χ_{10}	3	3ζ	$3\zeta^2$	0	0	0	0	0	0	0	0
χ_{11}	3	$3\zeta^2$	3ζ	0	0	0	0	0	0	0	0

Bringing together all character values from above, we obtain the entire character table, seen in Table 1.1.

(b) Extra Credit: Compute the character table of $H(\mathbb{F}_p)$ for larger primes

We will compute the character table for $H(\mathbb{F}_5)$. Put $G = H(\mathbb{F}_5)$. The conjugacy classes are similar as in the case of $p = 3$, there are just more of them. Using the same notation as above, we will denote the conjugacy classes by

$$C_0, \ldots, C_4, C_{01}, \ldots, C_{04}, \ldots\ldots, C_{40}, \ldots, C_{44}$$

size:	1	1	1	3	3	3	3	3	3	3	3
class:	C_0	C_1	C_2	C_{01}	C_{02}	C_{10}	C_{11}	C_{12}	C_{20}	C_{21}	C_{22}
χ_1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	ζ	ζ	ζ	ζ^2	ζ^2	ζ^2
χ_3	1	1	1	ζ	ζ^2	1	ζ	ζ^2	1	ζ	ζ^2
χ_4	1	1	1	ζ	ζ^2	ζ	ζ^2	1	ζ^2	1	ζ
χ_5	1	1	1	ζ^2	ζ	ζ	1	ζ^2	ζ^2	ζ	1
χ_6	1	1	1	ζ	ζ^2	ζ^2	1	ζ	ζ	ζ^2	1
χ_7	1	1	1	ζ^2	ζ	ζ^2	ζ	1	ζ	1	ζ^2
χ_8	1	1	1	ζ^2	ζ	1	ζ^2	ζ	1	ζ^2	ζ
χ_9	1	1	1	1	1	ζ^2	ζ^2	ζ^2	ζ	ζ	ζ
χ_{10}	3	3ζ	$3\zeta^2$	0	0	0	0	0	0	0	0
χ_{11}	3	$3\zeta^2$	3ζ	0	0	0	0	0	0	0	0

Table 1.1: Character table of $H(\mathbb{F}_3)$

Therefore there are 29 irreducible characters. The degrees of each of these characters must divide $|G| = 125$, i.e. be 1, 5, 10, or 25, and the sum of their squares must be 29. The only set of values that satisfy such contraints are 25 1's and 4 5's. This gives us the first column, and of course we have all 1's in the first row for the trivial representation. Now if we take φ and ϕ to be as they are defined in the previous subsection, the quotients of G by their respective kernels each result in $\mathbb{F}_5$. Thus each 1-degree character is made up of fifth roots of unity, which we will denote by ζ. Hence we can set χ_2 to the character arising from the character on $\mathbb{F}_5$ which takes $i \mapsto \zeta^i$, and similarly with χ' and ϕ. This is similar to our method above. We can then set the rest of the degree-1 characters to be, i.e. alterating "tensoring" by χ_2 and χ_3 so that

$$
\begin{aligned}
\chi_3 &= \chi_2^2 \\
\chi_4 &= \chi_2^3 \\
\chi_5 &= \chi_3^4 \\
\chi_6 &= \chi' \\
\chi_7 &= \chi_2\chi' \\
\chi_8 &= \chi_2^2\chi' \\
\chi_9 &= \chi_2^3\chi' \\
\chi_{10} &= \chi_2^4\chi' \\
\chi_{11} &= \chi'^2 \\
\chi_{12} &= \chi_2\chi'^2 \\
\chi_{13} &= \chi_2^2\chi'^2 \\
\chi_{14} &= \chi_2^3\chi'^2 \\
\chi_{15} &= \chi_2^4\chi'^2 \\
\chi_{16} &= \chi'^3 \\
\chi_{17} &= \chi_2\chi'^3 \\
\chi_{18} &= \chi_2^2\chi'^3 \\
\chi_{19} &= \chi_2^3\chi'^3 \\
\chi_{20} &= \chi_2^4\chi'^3 \\
\chi_{21} &= \chi'^4 \\
\chi_{22} &= \chi_2\chi'^4 \\
\chi_{23} &= \chi_2^2\chi'^4 \\
\chi_{24} &= \chi_2^3\chi'^4 \\
\chi_{25} &= \chi_2^4\chi'^4
\end{aligned}
$$

Now the remaining characters will be zero on any classes which are not one of the C_i classes. Furthermore, they will consist of 5's on the left and shifts of $(\zeta, \zeta^2, \zeta^3, \zeta^4)$. This results in table 1.2.

size:	1	1	1	1	1	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
class:	C_0	C_1	C_2	C_3	C_4	C_{01}	C_{02}	C_{03}	C_{04}	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{30}	C_{31}	C_{32}	C_{33}	C_{34}	C_{40}	C_{41}	C_{42}	C_{43}	C_{44}
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	ζ	ζ^2	ζ^3	ζ^4	1	ζ	ζ^2	ζ^3	ζ^4	1	ζ	ζ^2	ζ^3	ζ^4	1	ζ	ζ^2	ζ^3	ζ^4	1	ζ	ζ^2	ζ^3	ζ^4
χ_3	1	1	1	1	1	ζ^2	ζ^4	ζ	ζ^3	1	ζ^2	ζ^4	ζ	ζ^3	1	ζ^2	ζ^4	ζ	ζ^3	1	ζ^2	ζ^4	ζ	ζ^3	1	ζ^2	ζ^4	ζ	ζ^3
χ_4	1	1	1	1	1	ζ^3	ζ	ζ^4	ζ^2	1	ζ^3	ζ	ζ^4	ζ^2	1	ζ^3	ζ	ζ^4	ζ^2	1	ζ^3	ζ	ζ^4	ζ^2	1	ζ^3	ζ	ζ^4	ζ^2
χ_5	1	1	1	1	1	ζ^4	ζ^3	ζ^2	ζ	1	ζ^4	ζ^3	ζ^2	ζ	1	ζ^4	ζ^3	ζ^2	ζ	1	ζ^4	ζ^3	ζ^2	ζ	1	ζ^4	ζ^3	ζ^2	ζ
χ_6	1	1	1	1	1	1	1	1	1	ζ	ζ	ζ	ζ	ζ	ζ^2	ζ^2	ζ^2	ζ^2	ζ^2	ζ^3	ζ^3	ζ^3	ζ^3	ζ^3	ζ^4	ζ^4	ζ^4	ζ^4	ζ^4
χ_7	1	1	1	1	1	ζ	ζ^2	ζ^3	ζ^4	ζ	ζ^2	ζ^3	ζ^4	1	ζ^2	ζ^3	ζ^4	1	ζ	ζ^3	ζ^4	1	ζ	ζ^2	ζ^4	1	ζ	ζ^2	ζ^3
χ_8	1	1	1	1	1	ζ^2	ζ^4	ζ	ζ^3	ζ	ζ^3	1	ζ^2	ζ^4	ζ^2	ζ^4	ζ	ζ^3	1	ζ^3	1	ζ^2	ζ^4	ζ	ζ^4	ζ	ζ^3	1	ζ^2
χ_9	1	1	1	1	1	ζ^3	ζ	ζ^4	ζ^2	ζ	ζ^4	ζ^2	1	ζ^3	ζ^2	1	ζ^3	ζ	ζ^4	ζ^3	ζ	ζ^4	ζ^2	1	ζ^4	ζ^2	1	ζ^3	ζ
χ_{10}	1	1	1	1	1	ζ^4	ζ^3	ζ^2	ζ	ζ	1	ζ^4	ζ^3	ζ^2	ζ^2	ζ	1	ζ^4	ζ^3	ζ^3	ζ^2	ζ	1	ζ^4	ζ^4	ζ^3	ζ^2	ζ	1
χ_{11}	1	1	1	1	1	1	1	1	1	ζ^2	ζ^2	ζ^2	ζ^2	ζ^2	ζ^4	ζ^4	ζ^4	ζ^4	ζ^4	ζ	ζ	ζ	ζ	ζ	ζ^3	ζ^3	ζ^3	ζ^3	ζ^3
χ_{12}	1	1	1	1	1	ζ	ζ^2	ζ^3	ζ^4	ζ^2	ζ^3	ζ^4	1	ζ	ζ^4	1	ζ	ζ^2	ζ^3	ζ	ζ^2	ζ^3	ζ^4	1	ζ^3	ζ^4	1	ζ	ζ^2
χ_{13}	1	1	1	1	1	ζ^2	ζ^4	ζ	ζ^3	ζ^2	ζ^4	ζ	ζ^3	1	ζ^4	ζ	ζ^3	1	ζ^2	ζ	ζ^3	1	ζ^2	ζ^4	ζ^3	1	ζ^2	ζ^4	ζ
χ_{14}	1	1	1	1	1	ζ^3	ζ	ζ^4	ζ^2	ζ^2	1	ζ^3	ζ	ζ^4	ζ^4	ζ^2	1	ζ^3	ζ	ζ	ζ^4	ζ^2	1	ζ^3	ζ^3	ζ	ζ^4	ζ^2	1
χ_{15}	1	1	1	1	1	ζ^4	ζ^3	ζ^2	ζ	ζ^2	ζ	1	ζ^4	ζ^3	ζ^4	ζ^3	ζ^2	ζ	1	ζ	1	ζ^4	ζ^3	ζ^2	ζ^3	ζ^2	ζ	1	ζ^4
χ_{16}	1	1	1	1	1	1	1	1	1	ζ^3	ζ^3	ζ^3	ζ^3	ζ^3	ζ	ζ	ζ	ζ	ζ	ζ^4	ζ^4	ζ^4	ζ^4	ζ^4	ζ^2	ζ^2	ζ^2	ζ^2	ζ^2
χ_{17}	1	1	1	1	1	ζ	ζ^2	ζ^3	ζ^4	ζ^3	ζ^4	1	ζ	ζ^2	ζ	ζ^2	ζ^3	ζ^4	1	ζ^4	1	ζ	ζ^2	ζ^3	ζ^2	ζ^3	ζ^4	1	ζ
χ_{18}	1	1	1	1	1	ζ^2	ζ^4	ζ	ζ^3	ζ^3	1	ζ^2	ζ^4	ζ	ζ	ζ^3	1	ζ^2	ζ^4	ζ^4	ζ	ζ^3	1	ζ^2	ζ^2	ζ^4	ζ	ζ^3	1
χ_{19}	1	1	1	1	1	ζ^3	ζ	ζ^4	ζ^2	ζ^3	ζ	ζ^4	ζ^2	1	ζ	ζ^4	ζ^2	1	ζ^3	ζ^4	ζ^2	1	ζ^3	ζ	ζ^2	1	ζ^3	ζ	ζ^4
χ_{20}	1	1	1	1	1	ζ^4	ζ^3	ζ^2	ζ	ζ^3	ζ^2	ζ	1	ζ^4	ζ	1	ζ^4	ζ^3	ζ^2	ζ^4	ζ^3	ζ^2	ζ	1	ζ^2	ζ	1	ζ^4	ζ^3
χ_{21}	1	1	1	1	1	1	1	1	1	ζ^4	ζ^4	ζ^4	ζ^4	ζ^4	ζ^3	ζ^3	ζ^3	ζ^3	ζ^3	ζ^2	ζ^2	ζ^2	ζ^2	ζ^2	ζ	ζ	ζ	ζ	ζ
χ_{22}	1	1	1	1	1	ζ	ζ^2	ζ^3	ζ^4	ζ^4	1	ζ	ζ^2	ζ^3	ζ^3	ζ^4	1	ζ	ζ^2	ζ^2	ζ^3	ζ^4	1	ζ	ζ	ζ^2	ζ^3	ζ^4	1
χ_{23}	1	1	1	1	1	ζ^2	ζ^4	ζ	ζ^3	ζ^4	ζ	ζ^3	1	ζ^2	ζ^3	1	ζ^2	ζ^4	ζ	ζ^2	ζ^4	ζ	ζ^3	1	ζ	ζ^3	1	ζ^2	ζ^4
χ_{24}	1	1	1	1	1	ζ^3	ζ	ζ^4	ζ^2	ζ^4	ζ^2	1	ζ^3	ζ	ζ^3	ζ	ζ^4	ζ^2	1	ζ^2	1	ζ^3	ζ	ζ^4	ζ	ζ^4	ζ^2	1	ζ^3
χ_{25}	1	1	1	1	1	ζ^4	ζ^3	ζ^2	ζ	ζ^4	ζ^3	ζ^2	ζ	1	ζ^3	ζ^2	ζ	1	ζ^4	ζ^2	ζ	1	ζ^4	ζ^3	ζ	1	ζ^4	ζ^3	ζ^2
χ_{26}	5	5ζ	$5\zeta^2$	$5\zeta^3$	$5\zeta^4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{27}	5	$5\zeta^2$	$5\zeta^3$	$5\zeta^4$	5ζ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	5	$5\zeta^3$	$5\zeta^4$	5ζ	$5\zeta^2$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	5	$5\zeta^4$	5ζ	$5\zeta^2$	$5\zeta^3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1.2: Character table of $H(\mathbb{F}_5)$

2

(a) Find a non-commutative group $G \subset S_7$ with 21 elements.

Since S_7 has elements of order 3 and of order 7, e.g. (1 2 3) and (1 2 3 4 5 6 7), and so has subgroups isomorphic to cyclic groups C_3 and C_7. We can take a semi-direct product of these two cyclic groups to obtain the desired group with 21 elements.

Since $\mathrm{Aut}(C_3)$ is isomorphic to C_2, the identity map is the only homomophism in $\mathrm{Aut}(C_3)^{C_7}$. Hence any $C_3 \rtimes C_7$ would just be isomorphic to the direct product, thereby making G commutative give that the components C_3 and C_7 are commutative. Therefore we must find a non-trivial homomorphism $\varphi : C_3 \to \mathrm{Aut}(C_7)$ which will then make $C_7 \rtimes_\varphi C_3$ into a semi-direct product. The automorphisms on C_7 are defined by their mapping of 1, so define $f_n \in \mathrm{Aut}(C_7)$ by the map $m \mapsto nm$ for $m \in C_7$. Because φ needs to be a homomorphism, we must have

$$\begin{aligned}
\varphi(0) &= f_1 \\
\varphi(1) &= f_n \\
\varphi(2) &= \varphi(1+1) = f_n \circ f_n = f_{n^2}
\end{aligned}$$

for some yet to be determined $f_n \in \mathrm{Aut}(C_7)$. However, notice that no matter what f_n is, $\varphi(x) = f_{n^x}$ for any $x \in C_3$. Therefore $\varphi(x)(m) = f_{n^x}(m) = n^x m$, which in turn yields the composition of this semi-direct product:

$$(m_1, x_1)(m_2, x_2) = (m_1 + \varphi(x_1)(m_2), x_1 + x_2) = (m_1 + n^{x_1} m_2, x_1 + x_2) \tag{2.1}$$

For our group we will select $n = 2$

(b) Determine the conjugacy classes of G

With the fomula from equation 2.1 in hand, the conjugation of $(r, y) \in C_7 \rtimes_\varphi C_3$ by (m, x) is

$$\begin{aligned}
(m,x)(r,y)(\varphi(-x)(-m),\ -x) &= (m + 2^x r, x + y)(2^{-x}(-m),\ -x) \\
&= (m + 2^x r + 2^{x+y} 2^{-x}(-m),\ x + y - x) \\
&= (m + 2^x r - 2^y m,\ y) \\
&= (2^x r + (1 - 2^y)m,\ y)
\end{aligned}$$

Thus we have the following conjugates when conjugating by $(m, x) \in C_7 \rtimes_\varphi C_3$

$$\begin{array}{lll}
(0,0) \to (0,0) & (0,1) \to (-m,1) & (0,2) \to (-3m,2) \\
(1,0) \to (2^x,0) & (1,1) \to (2^x - m,1) & (1,2) \to (2^x - 3m,2) \\
(2,0) \to (2(2^x),0) & (2,1) \to (2(2^x) - m,1) & (2,2) \to (2(2^x) - 3m,2) \\
(3,0) \to (3(2^x),0) & (3,1) \to (3(2^x) - m,1) & (3,2) \to (3(2^x) - 3m,2) \\
(4,0) \to (4(2^x),0) & (4,1) \to (4(2^x) - m,1) & (4,2) \to (4(2^x) - 3m,2) \\
(5,0) \to (5(2^x),0) & (5,1) \to (5(2^x) - m,1) & (5,2) \to (5(2^x) - 3m,2) \\
(6,0) \to (6(2^x),0) & (6,1) \to (6(2^x) - m,1) & (6,2) \to (6(2^x) - 3m,2)
\end{array}$$

from which we obtain the conjugacy classes:

$$\begin{aligned}
C_0 &= \{(0,0)\} \\
C_{01} &= \{(3,0),(5,0),(6,0)\} \\
C_{02} &= \{(1,0),(2,0),(4,0)\} \\
C_1 &= \{(0,1),(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)\} \\
C_2 &= \{(0,2),(1,2),(2,2),(3,2),(4,2),(5,2),(6,2)\}
\end{aligned}$$

(c) Compute the character table for G

We will denote the characters by χ_1 through χ_5 with the first being the character for the trivial representation. Given this we have the following steps of reasoning about the values of the character table.

1. χ_1 is the trivial representation so

size:	1	3	3	7	7
class:	C_0	C_{01}	C_{02}	C_1	C_2
χ_1	1	1	1	1	1

2. Given the hint in the problem statement, there are three characters of degree 1. Since the degree of the characters need to divde the order of the group and the squares of their sums need to be 21 (in this case), the first column of the character table is $1, 1, 1, 3, 3$

3. Define the homomorphism $\phi : G \to C_3$ by $\phi(m, n) = m$. Then $\ker \phi = \{(m, 0) | n \in C_7\}$ implying that we can obtain degree-1 characters from the degree-1 characters of C_3. We will assign these to χ_2 and χ_3 yielding:

size:	1	3	3	7	7
class:	C_0	C_{01}	C_{02}	C_1	C_2
χ_2	1	1	1	ζ	ζ^2
χ_3	1	1	1	ζ^2	ζ

where ζ is the third root of unity, $e^{\frac{2\pi i}{3}}$.

4. Since $7(1 + \zeta\overline{\zeta} + \zeta^2\overline{\zeta^2}) = 7(3) = 21$, then on C_1 and C_2, χ_3 and χ_4 both have a value of zero.

5. Since $(2, 0)^{-1} = (\varphi(0)(5), 0) = (5, 0)$ and because $(5, 0) \in C_{01}$ while $(2, 0) \in C_{02}$, then the value of χ_4 on C_{01} and C_{02} must be conjugates of each other. By the same reasoning, the same is true for χ_5. If we let α be the value of χ_4 on C_{01}, pairing the fourth row with itself yields $\alpha\overline{\alpha} = 2$ and similarly if we let β be the value of χ_5 on C_{02}. Therefore the magnitude of α and β must be $\sqrt{2}$. We quickly realize that $\alpha = \sqrt{2}\zeta$ and $\beta = \sqrt{2}\zeta^2$ is a viable solution, yielding:

size:	1	3	3	7	7
class:	C_0	C_{01}	C_{02}	C_1	C_2
χ_4	3	$\sqrt{2}\zeta$	$\sqrt{2}\zeta^2$	0	0
χ_5	3	$\sqrt{2}\zeta^2$	$\sqrt{2}\zeta$	0	0

Therefore the whole character table for our G can be seen in table 2.3

size:	1	3	3	7	7
class:	C_0	C_{01}	C_{02}	C_1	C_2
χ_1	1	1	1	1	1
χ_2	1	1	1	ζ	ζ^2
χ_3	1	1	1	ζ^2	ζ
χ_4	3	$\sqrt{2}\zeta$	$\sqrt{2}\zeta^2$	0	0
χ_5	3	$\sqrt{2}\zeta^2$	$\sqrt{2}\zeta$	0	0

Table 2.3: Character table of $G = C_7 \rtimes_\varphi C_3$

(d) Extra Credit

3 Characters of symmetric and alternating products

Let (V, ρ) be a finite dimensional representation of a finite group G. Let χ_ρ be it's character. Fix an $x \in G$. The we can select a basis for V of eigenvectors of ρ_x, denote it by $\{e_i\}$ and the corresponding eigenvalues by $\{\lambda_i\}$. Then

$$\chi_\rho^2(x) = (\mathrm{Tr}(\rho_x))^2 = \left(\sum_i \lambda_i\right)^2 = \sum_{i,j} \lambda_i \lambda_j \tag{3.2}$$

and

$$\chi_\rho(x^2) = (\mathrm{Tr}\,(\rho_{x^2})) = \left(\mathrm{Tr}\,\left(\rho_x^2\right)\right) = \sum_i \lambda_i^2 \tag{3.3}$$

(a) Show that the character of $(S^2(V), S^2\rho)$ at $x \in G$ is $\frac{1}{2}\left(\chi_\rho^2(x) + \chi_\rho(x^2)\right)$

The elements $\{e_i \otimes e_j + e_j \otimes e_i\}_{i \leq j}$ form a basis for $S^2(V)$. Since

$$S^2\rho_x(e_i \otimes e_j + e_j \otimes e_i) = S^2\rho_x(e_i \otimes e_j) + S^2\rho_x(e_j \otimes e_i) = \rho_x e_i \otimes \rho_x e_j + \rho_x e_j \otimes \rho_x e_i = \lambda_i \lambda_j \left(e_i \otimes e_j + e_j \otimes e_i\right)$$

then by equations 3.2 and 3.3

$$\begin{aligned}
2\chi_{S^2\rho}(x) &= 2\left(\sum_{i \leq j} \lambda_i \lambda_j\right) \\
&= 2\left(\sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j\right) \\
&= \left(\sum_i \lambda_i^2 + 2\sum_{i<j} \lambda_i \lambda_j\right) + \sum_i \lambda_i^2 \\
&= \left(\sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j + \sum_{i<j} \lambda_i \lambda_j\right) + \chi_\rho(x^2) \\
&= \left(\sum_i \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j + \sum_{j<i} \lambda_j \lambda_i\right) + \chi_\rho(x^2) \\
&= \sum_{i,j} \lambda_i \lambda_j + \chi_\rho(x^2) \\
&= \chi_\rho^2(x) + \chi_\rho(x^2)
\end{aligned}$$

yielding the desired equality.

(b) Show that the character of $(\Lambda^2(V), \Lambda^2\rho)$ at $x \in G$ is $\frac{1}{2}\left(\chi_\rho^2(x) - \chi_\rho(x^2)\right)$

The elements $\{e_i \otimes e_j - e_j \otimes e_i\}_{i<j}$ form a basis for $\Lambda^2(V)$. Since

$$\Lambda^2\rho_x(e_i \otimes e_j - e_j \otimes e_i) = \Lambda^2\rho_x(e_i \otimes e_j) - \Lambda^2\rho_x(e_j \otimes e_i) = \rho_x e_i \otimes \rho_x e_j - \rho_x e_j \otimes \rho_x e_i = \lambda_i \lambda_j \left(e_i \otimes e_j - e_j \otimes e_i\right)$$

then by equations 3.2 and 3.3

$$\begin{aligned}
\chi_{\Lambda^2\rho}(x) &= \sum_{i<j} \lambda_i \lambda_j \\
&= \frac{1}{2}\left(\sum_{i<j} \lambda_i \lambda_j + \sum_{i<j} \lambda_i \lambda_j\right) \\
&= \frac{1}{2}\left(\sum_{i<j} \lambda_i \lambda_j + \sum_{j<i} \lambda_j \lambda_i\right) \\
&= \frac{1}{2}\left(\sum_{i,j} \lambda_i \lambda_j - \sum_{i} \lambda_i\right) \\
&= \frac{1}{2}\left(\chi_\rho^2(x) - \chi_\rho(x^2)\right)
\end{aligned}$$

yielding the desired equality.

(c) Extra Credit

4

(a)

(b)

(c)

(d)

5 Extra Credit