Math 503: Abstract Algebra
Homework 6

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Since the conjugacy classes of S_4 are

$$\begin{gathered}\{(1)\}\\ \{(12),(13),(14),(23),(24),(34)\}\\ \{(123),(132),(124),(142),(134),(143),(234),(243)\}\\ \{(12)(34),(13)(24),(14)(23)\}\\ \{(1234),(1243),(1324),(1342),(1423),(1432)\}\end{gathered}$$

which we will represent by the elements (1), (12), (123), (12)(34), and (1234), respectively. Thus there are five irreducible characters of S_4. Since the degrees of the irreducible characters must be factors of $|S_4| = 24$, and their squares must sum to 24, then the only possible degrees are $1, 1, 2, 3, 3$. Letting χ_1 be the trivial character and χ_2 be the character resulting from the sign homomorphism, we immediately have

size:	1	6	8	3	6
class:	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	a_3	b_3	c_3	d_3
χ_4	3	a_4	b_4	c_4	d_4
χ_5	3	a_5	b_5	c_5	d_5

Note that becuase each conjugacy class, C, has the property that $x \in C$ implies $x^{-1} \in C$, then all values of the irreducible characters must be real, since $\chi(\sigma^{-1}) = \overline{\chi(\sigma)}$ for any $\sigma \in S_4$ and character χ. Furthermore, since each character value must be the sum of roots of unity, then each current unknown in the table above must be an integer. In light of this and the fact that

$$\sum_i \chi_i^2(\sigma) = {}^{|S_4|}/_{|C_\sigma|}$$

for any $\sigma \in S_4$ and associated conjugacy class C_σ we get the following four equations

$$a_3^2 + a_4^2 + a_5^2 = 2 \tag{1.1}$$
$$b_3^2 + b_4^2 + b_5^2 = 1 \tag{1.2}$$
$$c_3^2 + c_4^2 + c_5^2 = 6 \tag{1.3}$$
$$d_3^2 + d_4^2 + d_5^2 = 2 \tag{1.4}$$

which yields the possible solution sets

$$a_3, a_4, a_5 \rightarrow 0, \pm 1, \pm 1 \tag{1.5}$$
$$b_3, b_4, b_5 \rightarrow 0, 0, \pm 1 \tag{1.6}$$
$$c_3, c_4, c_5 \rightarrow \pm 1, \pm 1, \pm 2 \tag{1.7}$$
$$d_3, d_4, d_5 \rightarrow 0, \pm 1, \pm 1 \tag{1.8}$$

Equation 1.2, solutions sets 1.6, and by pairing columns (123) and (1) reveals $b_3 = -1$ and $b_4 = b_5 = 0$. Then by pairing (1) with (12), (1) with (1234), and (12) with (1234) we get

$$\begin{aligned} 2a_3 + 3a_4 + 3a_5 &= 0\\ 2d_3 + 3d_4 + 3d_5 &= 0\\ a_3d_3 + a_4d_4 + a_5d_5 &= -2 \end{aligned}$$

which, when considering equations 1.1, 1.4, 1.5, and 1.8, imply that either $a_3 = d_3 = 0$, $a_4 = d_5 = 1$, and $a_5 = d_4 = -1$ is the solution, or $a_3 = d_3 = 0$, $a_4 = d_5 = -1$, and $a_5 = d_4 = 1$ is the solution. We will just choose the former, as chosing one arbitrarily just dictates to which irreducible characters χ_4 and χ_5 will correspond. Finally, pairing columns (123) and (12)(34) demands that $c_3 = 2$ (remember $b_3 = -1$ and $b_4 = b_5 = 0$), and pairing columns (1) and (12)(34) therefore demands that $c_4 = c_5 = -1$. Hence the full character table of S_4 is revealed in Table 1.1.

size:	1	6	8	3	6
class:	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Table 1.1: Character table of S_4

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Let p be a prime, and $H(\mathbb{F}_p)$ be the Heisenberg group with p^3 elememts.

Lemma 2.1. *The center of $H(\mathbb{F}_p)$ is*

$$Z(H(\mathbb{F}_p)) = \left\{ \begin{pmatrix} 1 & 0 & b \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\}_{b \in \mathbb{F}_p}$$

and is, furthermore, the same as the commutator subgroup, $(H(\mathbb{F}_p))'$.

Proof. For any $a, b, c, x, y, z \in \mathbb{F}_p$

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & a & cx - az + b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

and therefore for any $\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$ to be in the center of $H(\mathbb{F}_p)$, $cx - az$ must be zero, which, since x, z can be anything in $\mathbb{F}_p$, means $a = c = 0$. Hence the center of $H(\mathbb{F}_p)$ is

$$Z(H(\mathbb{F}_p)) = \left\{ \begin{pmatrix} 1 & 0 & b \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\}_{b \in \mathbb{F}_p}$$

Furthermore, because

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & cx - az \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then the elements of the center coincide with the elements of the commutator subgroup. $\square$

(a) Show that $H(\mathbb{F}_p)$ has $p^2 - 1$ conjugacy classes not contained in its center.

The proof of part (b) of this problem implies that there are $p^2 + p - 1$ total irreducible characters since $p^2(1^2) + (p-1)(p^2) = p^3 = |H(\mathbb{F}_p)|$, and it therefore has just as many conjugacy classes. Since each element of the center is the only element of its conjugacy class and the center has order p, there are then $(p^2 + p - 1) - p = p^2 - 1$ conjugacy classes with no intersection with the center.

(b) Show that $H(\mathbb{F}_p)$ has p^2 1-dimensional characters and $p-1$ irreducible characters of degree p

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By Lemma 2.1, there are

$$\left| {}^{H(\mathbb{F}_p)}/_{(H(\mathbb{F}_p))'} \right| = \left| {}^{H(\mathbb{F}_p)}/_{Z(H(\mathbb{F}_p))} \right| = {}^{p^3}/_{p} = p^2$$

one-dimensional irreducible characters.

Since the dimensions must both divide the order of the group, p^3, and their squares must sum to the order of the group, the dimensions are limited to being one or p since $(p^2)^2 > p^3$. Therefore Lemma 2.1 implies that there are n irreducible characters with degree p where n is the solution to $p^2(1^2) + n(p^2) = p^3$. Hence there are $n = p - 1$ p-dimensional irreducible characters.

(c) If χ is a non-abelian character, show that it vanishes outside of the center

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In light of Lemma 2.1, the abelianization of $H(\mathbb{F}_p)$ is $\mathbb{F}_p \times \mathbb{F}_p$, and we can therefore determine all the p^2 abelian characters by

$$\chi_{ij}\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} = \psi_i(a)\psi_j(c) \tag{2.9}$$

for $i, j \in \{0, \ldots, p-1\}$, where $\{\psi_i\}_{i \in \{0,\ldots,p-1\}}$ are the irreducible characters of $\mathbb{F}_p$. Fix a $g \in H(\mathbb{F}_p)$ with $g \notin Z(H(\mathbb{F}_p))$. Then each $\chi_{ij}(g)$ will be a p^{th} root of unity. Let g belong to the conjugacy class C. Note that $|C| = p$ as g is not in the center of $H(\mathbb{F}_p)$. Then

$$|C| \sum_{i,j} \chi_{ij}(g)\overline{\chi_{ij}(g)} = p \sum_{i,j} 1 = p^3 = |H(\mathbb{F}_p)|$$

implies $\chi(g)$ must be zero for any non-abelian character χ.

(d) Determine explicitly the character table of $H(\mathbb{F}_p)$

The first p^2 rows of the character table are the characters of χ_{ij} as in equation 2.9. The remaining characters are χ_1 through χ_{p-1} defined by $\chi_i = (\chi')^i$ where

$$\chi'(g) = \begin{cases} p\omega^n & g = A_n \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

where $\omega = e^{2\pi i/p}$ and

$$A_n = \begin{pmatrix} 1 & & n \\ & 1 & \\ & & 1 \end{pmatrix} \tag{2.11}$$

for $n \in \mathbb{F}_p$.

(e) Which characters are induced by proper subgroups.

Firstly, since the degree of an induced character is equal to the product of the degree of the character inducing it and the index of the inducing subgroup, then no 1-dimensional irreducible character of $H(\mathbb{F}_p)$ can be induced by a *proper* subgroup.

Now defining the subgroup $H \leqslant H(\mathbb{F}_p)$ to be the subgroup

$$\left\{ \begin{pmatrix} 1 & n & m \\ & 1 & \\ & & 1 \end{pmatrix} \middle| n, m \in \mathbb{F}_p \right\}$$

then it will have size p^2 and also be isomorphic to $\mathbb{F}_p \times \mathbb{F}_p$, i.e. we know it's character table. So define ψ to be the irreducible character of H by $\psi_{01} = \psi_0 \otimes \psi_1 \circ \alpha$ where $\alpha : H(\mathbb{F}_p) \to \mathbb{F}_p \times \mathbb{F}_p$ is the isomorphism

$$\begin{pmatrix} 1 & a & b \\ & 1 & \\ & & 1 \end{pmatrix} \mapsto (a, b)$$

ψ is the tirival character of $\mathbb{F}_p$, and ψ_1 is the character of $\mathbb{F}_p$ mapping $i \mapsto \omega^i$ where $\omega = e^{2\pi i/p}$. Since

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b + az - cx \\ & 1 & c \\ & & 1 \end{pmatrix}$$

then $\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01})$ will be zero on any element of $H(\mathbb{F}_p)$ which is not in H. Let $a \in \mathbb{F}_p$ be nonzero. Then since

$$\begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & a & b \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & b + az \\ & 1 & \\ & & 1 \end{pmatrix} \tag{2.12}$$

we see H is normal, and therefore z can be anything in $\mathbb{F}_p$

$$\begin{aligned}
\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01}) \begin{pmatrix} 1 & a & b \\ & 1 & \\ & & 1 \end{pmatrix} &= \frac{1}{|H|} \sum_{x \in H(\mathbb{F}_p)} \psi_{01}\left(x^{-1} \begin{pmatrix} 1 & a & b \\ & 1 & \\ & & 1 \end{pmatrix} x \right) \\
&= \frac{3}{p^2} \sum_{n \in \mathbb{F}_p} \psi_{01} \begin{pmatrix} 1 & a & n \\ & 1 & \\ & & 1 \end{pmatrix} \\
&= \frac{3}{p^2} \sum_{n \in \mathbb{F}_p} \psi_0(a)\psi_1(n) \\
&= \frac{3}{p^2} \sum_{n \in \mathbb{F}_p} \psi_1(n) \\
&= \frac{3}{p^2} \sum_{n \in \mathbb{F}_p} \omega^n \\
&= 0
\end{aligned}$$

implying that the induced representation of ψ_{01} is zero on elements of H outside the center. Combining this with the previous point about $\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01})$ being zero, we have that *any* element outside of the center has a value of zero under $\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01})$. now as for an element inside the center, equation 2.12 yields

$$\begin{aligned}
\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01}) \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} &= \frac{1}{|H|} \sum_{x \in H(\mathbb{F}_p)} \psi_{01}\left(x^{-1} \begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} x \right) \\
&= \frac{1}{p^2} \sum_{x \in H(\mathbb{F}_p)} \psi_0(0)\psi_1(b) \\
&= \frac{1}{p^2} \sum_{x \in H(\mathbb{F}_p)} \psi_1(b) \\
&= \frac{1}{p^2} \sum_{x \in H(\mathbb{F}_p)} \omega^b \\
&= \frac{1}{p^2} p^3 \omega^b \\
&= p\omega^b
\end{aligned}$$

Thus we see that $\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{01})$ is simply χ' from equation 2.10, and so ψ_{01} induces χ_1 defined in the previous part of the problem. Furthermore, by more generally defining $\psi_{ij} = \psi_i \otimes \psi \circ \alpha$ we see that

$$\begin{aligned}
\mathrm{Ind}_H^{H(\mathbb{F}_p)}(\psi_{0j})\begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix} &= \frac{1}{|H|}\sum_{x\in H(\mathbb{F}_p)} \psi_{0j}\left(x^{-1}\begin{pmatrix} 1 & & b \\ & 1 & \\ & & 1 \end{pmatrix}x\right) \\
&= \frac{1}{p^2}\sum_{x\in H(\mathbb{F}_p)} \psi_0(0)\psi_j(b) \\
&= \frac{1}{p^2}\sum_{x\in H(\mathbb{F}_p)} \psi_1^j(b) \\
&= \frac{1}{p^2}\sum_{x\in H(\mathbb{F}_p)} \omega^{jb} \\
&= \frac{1}{p^2}p^3\omega^{jb} \\
&= p\omega^{jb}
\end{aligned}$$

since $\psi_j = \psi_1^j$ for $1 \leq j \leq p-1$. Note that jb in the above equation is $\mod p$. Thus each ψ_{0j} induces χ_j as defined in the previous part of the problem. Hence each non-1-dimensional irreducible representation of $H(\mathbb{F}_p)$ is induced by a character of H.

3 Extra Credit

(a)

Since

$$\begin{pmatrix} x & y \\ & z \end{pmatrix}\begin{pmatrix} a & b \\ & c \end{pmatrix}\begin{pmatrix} x & y \\ & z \end{pmatrix}^{-1} = \begin{pmatrix} a & z^{-1}(bx + y(c-a)) \\ & c \end{pmatrix}$$

then when $a \neq c$ we have any

$$\begin{pmatrix} a & b \\ & c \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b' \\ & c \end{pmatrix}$$

are in the same conjugacy class. Furthermore, any two matrices whose diagonals do not match are in distinct conjugacy classes. Therefore, we denote by C_{ac}, the conjugacy class with diagonal a, c with $a \neq c$. Such matrices are a subset of the general linear group and therefore $a \neq 0$ and $c \neq 0$ for C_{ac}. Thus these conjugacy classes account for $(p-1)(p-2)$ of all the conjugacy classes.

The remaining conjugacy classes a made up of the matrices with the same entries along the diagonal. Now because

$$\begin{pmatrix} x & y \\ & z \end{pmatrix}\begin{pmatrix} a & b \\ & a \end{pmatrix}\begin{pmatrix} x & y \\ & z \end{pmatrix}^{-1} = \begin{pmatrix} a & z^{-1}bx \\ & a \end{pmatrix}$$

then $\begin{pmatrix} a & b \\ & a \end{pmatrix}$ will be in a conjugacy class by itself if $b = 0$, call it N_a, otherwise all elements of the above form are in the same conjugacy class, call it C_a. There are $p-1$ possible conjugacy classes N_a and likewise for C_a. Combining these counts with those above for C_{ac} we arrive a total of $2(p-1) + (p-1)(p-2) = p^2 - p$ conjugacy classes total.

Note that the center is made up of all the singleton classes N_a, accounting for $p-1$ elements, that each C_a has $p-1$ elements, and C_{ac} each has p elements.

(b)

(c)