Math 503: Abstract Algebra
Homework 7

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1 Cyclotomic Polynomials

(a)

(b)

(c)

(d) Extra Credit

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Let $K = \mathbb{Q}[T]/(T^3 - 2)$ and let x be the image of T in K.

(a) Show that K is an extension of degree 3 over $\mathbb{Q}$

The polynomial $T^3 - 2 \in \mathbb{Q}[T]$ is irreducible, and therefore K is generated over $\mathbb{Q}$ by $1, x, x^2$ and is therefore a vector space of degree $\deg(T^3 - 2) = 3$ over $\mathbb{Q}$.

(b)

Let α be a real number such that $\alpha^3 = 2$. Define $\overline{\tau}_j : \mathbb{Q}[T] \to \mathbb{C}$ for each $j \in \mathbb{Z}/3\mathbb{Z}$ by

$$\overline{\tau}_j(f(T)) = e^{2j\pi\sqrt{-1}/3} f(\alpha)$$

Therefore $T^3 - 2$ is contained within $\ker \overline{\tau}_j$ for each j. Since $K = \mathbb{Q}[T]/(T^3 - 2)$, this implies the existence of a unique ring homomorphism $\tau_j : K \to \mathbb{C}$ such that $\tau_j(\alpha) = \overline{\tau}_j(\alpha) = e^{2j\pi\sqrt{-1}/3}\alpha$

(c)

For γ_0, γ_1, and γ_2, let $\gamma_n = \sqrt[3]{2}e^{2\pi n\sqrt{-1}/3}$. Then since these are each roots of $T^3 - 2$ over $\mathbb{Q}$ then

$$K_j \cong K \cong \mathbb{Q}(\gamma_n)$$

for each n. Therefore $L \cong \mathbb{Q}(\gamma_0, \gamma_1, \gamma_2)$, however, we have the linear relation $\gamma_0 + \gamma_1 + \gamma_2 = 0$ due to each γ_n being distinct third roots of 2. This implies that $\mathbb{Q}(\gamma_0, \gamma_1, \gamma_2)$ is actually isomorphic to $\mathbb{Q}(\gamma_1, \gamma_2)$, without loss of generality in selecting two elements of $\gamma_0, \gamma_1, \gamma_2$, since any two are linearly independent. Hence, so is L. Now because $\mathbb{Q}(\gamma_1)$ and $\mathbb{Q}(\gamma_2)$ are have trivial intersection and each is generated by three elements over $\mathbb{Q}$, then L must be generated by six elements over $\mathbb{Q}$, i.e. $[L : \mathbb{Q}] = 6$.

(d) Show that there exists a surjective ring homomorphism $K \otimes_{\mathbb{Q}} K \to L$

(e)

3

Let L_1, L_2 be finite extensions of a field k with degrees 3 and 2, respectively.

(a)

Since the degrees of L_1 and L_2 over k are finite, then they are both algebraic over k. Letting $\alpha \in L_1 - k$, we have $[k(\alpha) : k]$ divides $[L_1 : k]$ and $[k(\alpha) : k] > 1$, but since $[L_1 : k] = 3$, i.e. $[L_1 : k]$ is prime, then $[k(\alpha) : k]$ must also be 3. Similarly for $\beta \in L_2 - k$, $[k(\beta) : k] = 2$. Therefore $L_1 \cong k(\alpha)$ and $L_2 \cong k(\beta)$, which in turn implies

$$L_1 \cong {}^{k[x]}/\big(m_\alpha(x)\big) \qquad \text{and} \qquad L_2 \cong {}^{k[x]}/\big(m_\beta(x)\big)$$

where $m_\alpha(x)$ is the minimal polynomial of α, and likewise for β and $m_\beta(x)$. This reveals $L_1 \otimes_k L_2 \cong {}^{k[x]}/\big(m_\alpha(x)\big) \otimes_k {}^{k[x]}/\big(m_\beta(x)\big)$, but because $m_\alpha(x)$ is irreducible over $k[x]$ of degree $[k(\alpha) : k] = 3$ and $m_\beta(x)$ is irreducible over $k[x]$ of degree $[k(\beta) : k] = 2$, then $m_\alpha(x)$ and $m_\beta(x)$ are coprime. Therefore

$$L_1 \otimes_k L_2 \cong {}^{k[x]}/\big(m_\alpha(x), m_\beta(x)\big)$$

but furthermore that $\big(m_\alpha(x), m_\beta(x)\big)$ is maximal, implying ${}^{k[x]}/\big(m_\alpha(x), m_\beta(x)\big)$ is a field and thus $L_1 \otimes_k L_2$ is as well.

(b)

(c)

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(a)

(b)

(c)