Math 503: Abstract Algebra
Homework 8

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(a)

Since $\cos(2\pi/5) = 1/4(\sqrt{5} - 1)$, then $\mathbb{Q}(\cos(2\pi/5))$ is simply $\mathbb{Q}(\sqrt{5})$, therefore being an extension of degree two since $x^2 - 5$ is the minimal polynomial of $\sqrt{5}$. Also since $\cos(2\pi/6) = \cos(\pi/3) = 1/2$ then $\mathbb{Q}(\cos(2\pi/6))$ is just $\mathbb{Q}$; and hence it is an extension of degree one.

(b) Extra Credit

2

(a) Is $\mathbb{Q}(\sqrt{3} + \sqrt{5})$

Because $\sqrt{3} + \sqrt{5} \in \mathbb{Q}(\sqrt{3}, \sqrt{5})$ then $\mathbb{Q}(\sqrt{3} + \sqrt{5}) \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Similarly, because

$$\frac{1}{4}\left((\sqrt{3} + \sqrt{5})^3 - 14(\sqrt{3} + \sqrt{5})\right) = \sqrt{3}$$

and

$$\frac{-1}{4}\left((\sqrt{3} + \sqrt{5})^3 - 18(\sqrt{3} + \sqrt{5})\right) = \sqrt{5}$$

then we also have $\mathbb{Q}(\sqrt{3}, \sqrt{5}) \subseteq \mathbb{Q}(\sqrt{3} + \sqrt{5})$. Putting these two results together indicates that $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ and $\mathbb{Q}(\sqrt{3} + \sqrt{5})$ are one in the same.

(b) Extra Credit: Show that $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ has a finite number of subfields)

Any subfield $K \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{5})$ will be a subspace of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Since $\mathbb{Q}(\sqrt{3}, \sqrt{5})$ is a finite extension, it is therefore a finite vector space over $\mathbb{Q}$, and hence only has a finite number of vector subspaces. Thus there are only a finite number of subfields, too.

3

(a)

(b) Extra Credit

4

Let $f(x)$ be an irreducible polynomial over a field F. Let K be a finite extension field of F. Let $g(x)$ and $h(x)$ are two irreducible factors of $f(x)$ in $K[x]$.

(a)

Let $F = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-3})$, then the non trivial element σ of $\text{Aut}(K/Q)$ is defined by $\sigma(a + b\sqrt{-3}) = a - b\sqrt{-3}$ for all $a + b\sqrt{-3} \in K$. Since $\sigma_* : K[x] \to K[x]$ is defined by $\sigma_*(x) = x$ (where x is the indeterminate) and $\sigma_*|_K = \sigma$.

Since $f(x)$ is irreducible over $F[x]$ but reducible as $f(x) = g(x)h(x)$ over $K[x]$ where $g(x), h(x)$ are irreducible in $K[x]$, then it must be the case that either $g(x)$ or $h(x)$ has some term with a coefficient in $K - F$, otherwise $f(x)$ would be reducible in $F[x]$. Without loss of generality, let $g(x)$ have a term with coefficient in $K - F$. Since $f(x) \in F[x]$, then $\sigma_*(f(x)) = \sigma_*(g(x)h(x)) = (\sigma_* g)(x)(\sigma_* h)(x) = f(x)$. Because $g(x)$ divides $f(x)$, then it divides $(\sigma_* g)(x)(\sigma_* h)(x)$, but because $g(x)$ has a term $(a + b\sqrt{-3})x^i$ for nonzero $b \in F$, then $(\sigma_* g)(x)$ will have the term $(a - b\sqrt{-3})x^i$ and therefore not be equal to $g(x)$. Hence $g(x) = (\sigma_* h)(x)$.

(b) Extra Credit

(c) Extra Credit