Math 503: Abstract Algebra
Homework 9

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1

(a)

(b)

Assuming the equation

$$disc(f) = -4b^3 - 27c^2$$

holds for $f(T) = T^3 + bT + c \in F[T]$ then we can determine the formula for a cubic polynomial $g(T) = T^3 + AT^2 + BT + C \in F[T]$ by substituting $S - A/3$ for T [DF04]. So $g(S - A/3)$ yields a polynomial $h(S)$ so that

$$g(S - A/3) = h(S) = S^3 + PS + Q$$

where $P = 1/3(3B - A^2)$ and $Q = 1/27(2A^3 - 9AB + 27C)$. Now if α, β, and γ are roots of $h(S)$, then $\alpha - A/3$, $\beta - A/3$, and $\gamma - A/3$ are roots of $g(T)$ since, for instance, $g(\alpha - A/3) = h(\alpha) = 0$. Furthermore, since $[((\alpha - A/3) - (\beta - A/3))((\beta - A/3) - (\gamma - A/3))((\gamma - A/3) - (\alpha - A/3))]^2 = [(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)]^2$, then the discriminents of $g(T)$ and $h(S)$ are the same, i.e. we can use the formula from the previous part of the problem (again assuming it holds) for the discriminent of $h(S)$. Hence

$$\begin{aligned}
disc(g) &= disc(h) \\
&= -4P^3 - 27Q^2 \\
&= -4(1/3(3B - A^2))^3 - 27(1/27(2A^3 - 9AB + 27C))^2 \\
&= -4/9(-\frac{1}{27}A^6 + \frac{1}{3}A^4B - A^2B^2 + B^3) - 1/27(\frac{4}{729}A^6 - \frac{4}{81}A^4B + \frac{1}{9}A^2B^2 + \frac{4}{27}A^3C - \frac{2}{3}ABC + C^2) \\
&= -4A^3C + A^2B^2 + 18ABC - 4B^3 - 27C^2
\end{aligned}$$

(c)

(d)

2

(a)

(b)

(c)

3

Exact values of $\cos(2\pi/5)$ and $\sin(2\pi/5)$ were computed with the help of Wolfram Alpha, [Wol14a], [Wol14b].

(a) Show that $\cos(2\pi/5)$ and $\sin(2\pi/5)$ are algebraic over $\mathbb{Q}$

Since $\cos(2\pi/5) = 1/4(\sqrt{5}-1)$ then $4\cos(2\pi/5)+1 = \sqrt{5}$, which implies $(4\cos(2\pi/5)+1)^2 = 16\cos^2(2\pi/5) + 8\cos(2\pi/5)+1$ is 5. Therefore $16\cos^2(2\pi/5)+8\cos(2\pi/5)-4$ is zero. So $\cos(2\pi/5)$ is algebraic since it is the root of

$$4x^2+2x-1 \in \mathbb{Q}[x] \tag{3.1}$$

since $16x^2+8x-4 = 4(4x^2+2x-1)$.

Since $\sin(2\pi/5) = \sqrt{5/8+\frac{\sqrt{5}}{8}}$, then $8\sin^2(2\pi/5)-5 = \sqrt{5}$, which implies that $64\sin^4(2\pi/5)-80\sin^2(2\pi/5)+25$ is 5. Subtracting five yields $64\sin^4(2\pi/5)-80\sin^2(2\pi/5)+20 = 0$, and thus that $\sin(2\pi/5)$ is algebraic since it is the root of

$$16x^4-20x^2+5 \in \mathbb{Q}[x] \tag{3.2}$$

since $64x^4-80x^2+20 = 4(16x^4-20x^2+5)$.

(b) Determine the degrees $[\mathbb{Q}(\cos(2\pi/5) : \mathbb{Q}]$ and $[\mathbb{Q}(\sin(2\pi/5)) : \mathbb{Q}]$. Does one contain the other?

The polynomial, $4x^2+2x-1$, from equation 3.1 is irreducible over $\mathbb{Q}$ since it has non rational roots $\frac{-2\pm\sqrt{5}}{4}$. Thus $\mathbb{Q}(\cos(2\pi/5)) \cong \mathbb{Q}[x]/(4x^2+2x-1)$, implying $[\mathbb{Q}(\cos(2\pi/5)) : \mathbb{Q}] = 2$.

The polynomial, $16x^4-20x^2+5$, from equation 3.2 is irreducible over $\mathbb{Q}$ since it meets Eisenstein's criteria, that is $5 \not| 16$, $5|20$, $5|5$, and $5^2 \not| 5$. So $\mathbb{Q}(\sin(2\pi/5)) \cong \mathbb{Q}[x]/(16x^4-20x^2+5)$, and therefore $[\mathbb{Q}(\sin(2\pi/5)) : \mathbb{Q}] = 4$.

Does one contain the other? Since $\sin(2\pi/5) = \sqrt{5/8+\frac{\sqrt{5}}{8}}$, then

$$8\sin^2(2\pi/5)-5 = \sqrt{5} \tag{3.3}$$

Therefore, because $\cos(2\pi/5) = 1/4(\sqrt{5}-1)$ we have

$$\begin{aligned}\cos(2\pi/5) &= 1/4(\sqrt{5}-1)\\ &= 1/4((8\sin^2(2\pi/5)-5)-1)\\ &= 2\sin^2(2\pi/5-3/2\end{aligned}$$

and so $\cos(2\pi/5)$ can be written as a linear combination of $\sin(2\pi/5)$ over $\mathbb{Q}$. Hence $\mathbb{Q}(\cos(2\pi/5)) \subset \mathbb{Q}(\sin(2\pi/5))$ without equality since, as seen above, their degrees over $\mathbb{Q}$ are not the same.

(c) Determine the Galois groups of $\mathbb{Q}(\cos(2\pi/5))$ and $\mathbb{Q}(\sin(2\pi/5))$

Since the field $\mathbb{Q}(\cos(2\pi/5)) \cong \mathbb{Q}[x]/(4x^2+2x-1)$ contains both the roots $\pm\cos(2\pi/5)$ of $4x^2+2x-1$ then the field is Galois, as it is simply the splitting field of the separable polynomial $4x^2+2x-1$. Since the field is an extension of degree two over $\mathbb{Q}$, then it's Galois group is simply $\mathbb{Z}/2\mathbb{Z}$.

Galois group of $\mathbb{Q}(\sin(2\pi/5))$ Now attempting to determine the Galois group for $\mathbb{Q}(\sin(2\pi/5)) \cong \mathbb{Q}[x]/(16x^4-20x^2+5)$ is slightly more tricky. Certainly the roots $\pm\sin(2\pi/5) = \pm\sqrt{5/8+\sqrt{5}/8}$ of $16x^4-20x^2+5$ are contained within, but it is not immediately clear whether or not the roots $\pm\sqrt{5/8-\sqrt{5}/8}$ are contained in the field. We note that

$$\sin(2\pi/5)\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}} = \sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}} = \frac{2\sqrt{5}}{8}$$

which implies that

$$\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}} = \frac{2\sqrt{5}}{8}(\sin(2\pi/5))^{-1}$$

Equation 3.3 informs us that $\sqrt{5}$ is in $\mathbb{Q}(\sin(2\pi/5))$, and being a field, $(\sin(2\pi/5))^{-1}$ is in definitely in $\mathbb{Q}(\sin(2\pi/5))$; hence $\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$ (and its negation) is also in the field. Hence all the roots of $16x^4 - 20x^2 + 5$ are contained within the field. Thus the field is therefore a splitting field of a separable polynomial, and hence Galois.

Now since $\mathbb{Q}(\sin(2\pi/5))$ has degree four over $\mathbb{Q}$, then its Galois group has size four. Automorphisms of the Galois group will just permute the roots $\pm\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$, $\pm\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$, because of which we can completely define two non-trivial automorphisms by

$$\sigma = \begin{cases} s & \mapsto & -s \\ t & \mapsto & t \end{cases} \qquad \text{and} \qquad \tau = \begin{cases} s & \mapsto & s \\ t & \mapsto & -t \end{cases}$$

letting $s = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}$ and $t = \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$. From this we see that σ^2 and τ^2 are both the identity automorphism and that the the third non-trivial automorphism is

$$\sigma\tau = \begin{cases} s & \mapsto & -s \\ t & \mapsto & -t \end{cases}$$

also with $(\sigma\tau)^2$ being the identity automorphism. Hence $\mathrm{Gal}(\mathbb{Q}(\sin(2\pi/5))/\,\mathbb{Q}) = \{1, \sigma, \tau, \sigma\tau\}$, but futhermore, since each non-trivial element has order two, then $\mathrm{Gal}(\mathbb{Q}(\sin(2\pi/5))/\,\mathbb{Q})$ must be the Klein-four group.

4 Extra Credit

(a)

(b)

(c)

References

[DF04] D.S. Dummit and R.M. Foote. *Abstract Algebra*. John Wiley & Sons Canada, Limited, 2004.

[Wol14a] Wolfram—Alpha. `http://www.wolframalpha.com/input/?i=cos(2*pi/5)`, April 2014.

[Wol14b] Wolfram—Alpha. `http://www.wolframalpha.com/input/?i=sin(2*pi/5)`, April 2014.