Math 503: Abstract Algebra
Homework 11

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May 7, 2014
http://coursework.tylerlogic.com/courses/upenn/math503/homework11

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Given the Fundamental Theorem of Galois Theory, in order to find fields $F \subset K \subset L$ such that K/F and L/K are finite Galois, but L/F is not Galois, it suffices to find a finite Galois extension M/F with Galois group G with subgroups H_1, H_2 such that $H_1 \trianglelefteq H_2$ and $H_2 \trianglelefteq G$, but $H_1 \ntrianglelefteq G$. In which case, we can put $K = M^{H_2}$ and $L = M^{H_1}$ in order to achieve the desired result.

In the last homework we saw that the splitting field of $T^4 - 3$ over $\mathbb{Q}$ is finite Galois with Galois group D_8. In the group D_8 we have subgroups $\langle sr \rangle \leqslant \langle sr, r^2 \rangle \leqslant D_8$ where s is the mirror symmetry and r is the rotational symmetry.

The fact that $r(sr)r^3 = rsr^4 = rs = sr^3$ demonstrates that $\langle sr \rangle$ is not normal in D_8. Since $\langle sr, r^2 \rangle = \{1, r^2, sr, sr^3\}$, then

$$\begin{aligned} r^2(sr)(r^2)^{-1} &= r^2 srr^2 = r^2 sr^3 = sr \\ sr(sr)(sr)^{-1} &= srsrsr = sr \\ sr^3(sr)(sr^3)^{-1} &= sr^3 srrs = sr^3 sr^2 s = sr \end{aligned}$$

shows that $\langle sr \rangle \trianglelefteq \langle sr, r^2 \rangle$, and

$$\begin{aligned} s^m r^n (r^2) r^{-n} s^m &= s^m r^2 s^m = s^m s^m r^{-2} = r^2 \\ s^m r^n (sr) r^{-n} s^m &= s^m r^n sr^{1-n} s^m = s^{m+1} r^{1-2n} s^m = s^{2m+1} r^{2n+1} = sr^{2n+1} \\ s^m r^n (sr^3) r^{-n} s^m &= s^m r^n sr^{3-n} s^m = s^{m+1} r^{3-2n} s^m = s^{2m+1} r^{2n-3} = sr^{2(n-1)+1} \end{aligned}$$

shows that $\langle sr, r^2 \rangle \trianglelefteq D_8$.

Thus for $F = \mathbb{Q}$, $K = \Omega^{\langle sr, r^2 \rangle}$ and $L = \Omega^{\langle sr \rangle}$ where $\Omega = \mathbb{Q}(\sqrt[4]{3}, i)$ is the splitting field of $T^4 - 3$ over $\mathbb{Q}$, we will have our desired scenario.

We can determine the fixed fields K and L as follows. Define s and r as the automorphisms

$$r = \begin{cases} \theta & \mapsto & i\theta \\ i & \mapsto & i \end{cases} \qquad\qquad s = \begin{cases} \theta & \mapsto & \theta \\ i & \mapsto & -i \end{cases}$$

where $\theta = \sqrt[4]{3}$. This then sets

$$sr = \begin{cases} \theta & \mapsto & -i\theta \\ i & \mapsto & -i \end{cases} \qquad\qquad r^2 = \begin{cases} \theta & \mapsto & -\theta \\ i & \mapsto & i \end{cases}$$

Because $\langle sr \rangle$ is a subgroup of size two, then $L/\mathbb{Q}$ must be an extension of degree four. Since

$$sr(\theta(i-1)) = sr(\theta)(sr(i) - 1) = -i\theta(-i-1) = \theta(i-1)$$

and $\mathbb{Q}(\theta(i-1)) \subset \Omega$ is an extension of degree four, then we must have $L = \mathbb{Q}(\theta(i-1))$. Similarly, $\langle sr, r^2 \rangle$ is a subgroup of size four, and $\mathbb{Q}(i\theta^2) \subset \mathbb{Q}(\theta(i-1)) \subset \Omega$ has degree two. Then since

$$sr(i\theta^2) = sr(i)(sr(\theta))^2 = -i(-i\theta)(-i\theta) = i\theta^2$$

and

$$r^2(i\theta^2) = r^2(i)(r^2(\theta))^2 = i(-\theta)^2 = i\theta^2$$

then we have $K = \mathbb{Q}(i\theta^2)$.

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Let L be a finite Galois extension of $\mathbb{R}$.

(a) Prove $\mathrm{Gal}(L/\mathbb{R})$ is a 2-group

Lemma 3.1. *Any non-trivial finite extension of $\mathbb{R}$ has even degree.*

Proof. If K is a non-trivial finite extension of $\mathbb{R}$, then it is algebraic over $\mathbb{R}$. For any $\alpha \in K - \mathbb{R}$, it's minimal polynomial $m_\alpha(x) \in \mathbb{R}[x]$ is nonlinear, irreducible. Since all odd degree polynomials in $\mathbb{R}[x]$ have a real root, we conclude $\deg(m_\alpha(x))$ is even. However, $[K : \mathbb{R}] = \deg(\alpha) = \deg(m_\alpha(x))$ and thus K has even degree. $\square$

By the preceeding lemma, $[L : \mathbb{R}]$ is even. Then so is the size of $\mathrm{Gal}(L/\mathbb{R})$. So define H to be a 2-Sylow subgroup of $\mathrm{Gal}(L/\mathbb{R})$. Then the index of H in $\mathrm{Gal}(L/\mathbb{R})$ is not divisible by two and therefore $E = L^H$ is an extension of $\mathbb{R}$ with odd degree. However, according to Lemma 3.1, the only such extension is $\mathbb{R}$ itself. Thus $E = \mathbb{R}$. This furthermore implies that $H = \mathrm{Gal}(L/\mathbb{R})$, which, since H is a 2-Sylow subgroup, also means $\mathrm{Gal}(L/\mathbb{R})$ is a 2-group.

(b) Prove $\mathbb{C}$ is algebraic

Lemma 3.2. *There is no irreducible quadratic over $\mathbb{C}$*

Proof. Any quadratic over $\mathbb{C}$ has roots in $\mathbb{C}$ provided by the quadratic formula. $\square$

Let $f(T) \in \mathbb{C}[T]$ and α be a root of $f(T)$. Assume by way of contradiction that $\mathbb{C}(\alpha)$ is a nontrivial extension over $\mathbb{C}$. Then $\mathbb{C}(\alpha)$ is an extension of $\mathbb{R}$ of even degree by Lemma 3.1. Thus, $\mathrm{Gal}(\mathbb{C}(\alpha)/\mathbb{R})$ would be a nontrivial 2-group with size greater than or equal to 4, by part (a) of this problem. But then $\mathrm{Gal}(\mathbb{C}(\alpha)/\mathbb{R})$ would have a subgroup of size 4 (Theorem 6.1 [DF04]: "p-groups have subgroups of all applicable sizes"), i.e. there would exist an extension $K/\mathbb{C}$ with degree 2. However, this contradicts the Lemma 3.2, and hence $\mathbb{C}(\alpha)$ is trivial, implying $\alpha \in \mathbb{C}$.

4 Extra Credit

5 Extra Credit

References

[DF04] D.S. Dummit and R.M. Foote. *Abstract Algebra.* John Wiley & Sons Canada, Limited, 2004.