Math 508: Advanced Analysis
Homework 2

Lawrence Tyler Rush
<me@tylerlogic.com>

September 12, 2014
http://coursework.tylerlogic.com/courses/upenn/math508/homework02

1 Density of rationals in reals

(a) For $a \in \mathbb{Q}$ with $a > 0$ and $a^2 < 2$ find rational $b > a$ with $a^2 < b^2 < 2$

First, if $a < 1$, then we can simply let $b = 1$ since $a^2 < 1^2 < 2$ in this case. So assume that $a \geq 1$. This implies $2a + 1 \geq 3$. Furthermore, this implies that $2 - a^2 < 2 - a < 2$, from which we conclude $2 - a^2 < 2a + 1$. Thus, defining $t = \frac{2-a^2}{2a+1}$, means that $0 < t < 1$.

With this definition of t, setting $b = a + t$ we see $b^2 = (a + t)^2 = a^2 + 2at + t^2 < a^2 + 2at + t$ where the last inequality comes from the fact that $0 < t < 1$. Continuing, we have

$$b^2 < a^2 + t(2a + 1) = a^2 + \frac{2 - a^2}{2a + 1}(2a + 1) = a^2 + 2 - a^2 = 2$$

Thus since $b = a+t$ and both a and t are positive, we have $a^2 < b^2 < 2$. Finally, t is rational since it is constructed by a combination of multiplication and addition of rational numbers; this implies that rationality of b since it is the sum of two rationals, a and t.

(b) For $c \in \mathbb{Q}$ with $c > 0$ and $2 < c^2$, find rational $d > 0$ with $d < c$ and $2 < d^2 < c^2$

Define $t = \frac{c^2-2}{2c}$. Since $0 < c < 2 < c^2$ then

$$t > 0 \qquad \text{and} \qquad c - t = c - \frac{c^2 - 2}{2c} = \frac{4c^2 - c^2 + 2}{2c} = \frac{3c^2 + 2}{2c} > 0$$

Now let $d = c - t$. The above equations imply that $0 < d < c$ and also allow for the use of the inequality in the following equation.

$$(c - t)^2 = c^2 - 2ct + t^2 < c^2 - 2ct = c^2 - 2c\frac{c^2 - 2}{2c} = c^2 - c^2 + 2 = 2$$

Given this equation, we now have $2 < d^2 < c^2$, and furthermore, since d is equal to some combination of rational numbers which are added/multiplied together, then it too is rational.

(c)

2 Some properties of an ordered field.

Let F be an ordered field containing elements x and y.

(a) Show that $x < y$ implies $x < \frac{x+y}{2} < y$

Let $x < y$. Then the definition of an ordered field and Rudin's Proposition 1.18 yield $x + x < x + y$, implying $2x < x + y$ and thus $x < \frac{x+y}{2}$. Similarly we have obtain $x + y < y + y$, $x + y < 2y$, and $\frac{x+y}{2} < y$. Combining these results we have

$$x < \frac{x + y}{2} < y$$

as desired.

(b) Prove $x \neq 0$ implies $x^2 > 0$

First suppose $x > 0$. Then Rudin's Proposition 1.18 (b) implies $x(x) > x(0)$ which is equivalent to $x^2 > 0$.

Now suppose $x < 0$. Rudin's Proposition 1.18 (c) implies $x(x) > x(0)$, i.e. $x^2 > 0$.

(c) Prove $x^2 + y^2 = 0$ implies $x = y = 0$

Lemma 2.1. *If F is some ordered field and $a, b \in F$ are such that $a > 0$ and $b > 0$, then $a + b > 0$.*

Proof. Let F be an ordered field with $a, b \in F$ such that $a > 0$ and $b > 0$. Adding b to the both sides of $a > 0$ yields $a + b > b$, but $b > 0$, so $a + b > 0$. □

Let x, y not be both identically zero. Without loss of generality, assume $x \neq 0$. The previous problem then implies $x^2 > 0$. So if $y = 0$, then $x^2 + y^2 = x^2 > 0$. If $y \neq 0$, then the previous problem gives $y^2 > 0$ which the above lemma then yields $x^2 + y^2 > 0$. Hence, in any case, $x^2 + y^2 \neq 0$.

(d) Show that $2xy \leq x^2 + y^2$. When does equality occur?

Part (b) of this problem implies that for any x, y we have $(x - y)^2 \geq 0$. Thus $x^2 - 2xy + y^2 \geq 0$, and therefore $x^2 + y^2 \geq 2xy$, as desired.

When does equality occur? Stepping backwards through this proof, we see that $x^2+y^2 = 2xy$ when $(x-y)^2 = 0$. Then part (b) of this problem implies that $x - y = 0$. Thus we have equality when $x = y$.

3 More rational density

(a) Show there is an irrational between rationals $x < y$

If $x < y$, then $0 < y-x$. Hence the archimedean property of the reals yields an integer $n > 0$ such that $n(y-x) > \sqrt{2}$. Since $\sqrt{2} > 0$ we have

$$\begin{aligned} 0 <& \quad \sqrt{2} < n(y-x) \\ 0 <& \quad \sqrt{2}/n < y - x \\ x <& x + \sqrt{2}/n < y \end{aligned}$$

Now we saw in the last homework that both the multiplication and sum of a rational with an irrational is irrational, so $x + \sqrt{2}/n$ is irrational since $\sqrt{2}$ is irrational.

(b) Show there is a rational between any real numbers $x < y$

If $x < y$, then $0 < y - x$. So again, the archimedean property gives us an integer $n > 0$ such that $ny - nx > 1$. This implies, since consecutive integers have a difference of one, that there must some integer m with $nx < m < ny$. Thus $x < m/n < y$, and m/n is rational.

4

5

(a) Find all sets $A \subset \mathbb{R}$ such that $\sup A \leq \inf A$

Let $A \subset \mathbb{R}$ with $a = \inf A$ and $b \sup A$. If $x, y \in A$ and $x < y$ then we would have $a \leq x < y \leq b$, and so it's not possible for a set with two or more elements to have a supremum that's less than or equal to the infimum. Hence only singleton sets have the desired property.

(b) If $A \subset \mathbb{R}$ is bounded above and $B \subset \mathbb{R}$ is bounded below, prove $A \cap B$ is bounded.

Let $A \subset \mathbb{R}$ be bound above by α and $B \subset \mathbb{R}$ be bounded below by β. Then $a \leq \alpha$ for all $a \in A$ and $\beta \leq b$ for all $b \in B$, however, since $A \cap B \subset A$ and $A \cap B \subset B$, then we must have that $\beta \leq x \leq \alpha$ for all $x \in A \cap B$. In other words, $A \cap B$ is bounded.

6

Let $z, w, v \in \mathbb{C}$ be complex numbers.

(a) Prove $|z - w| \geq |z - v| - |v - w|$.

We know for $a, b \in \mathbb{C}$ that $|a + b| \leq |a| + |b|$, which implies $|a + b| - |b| \leq |a|$. Since a and b are arbitrary, then we can find $x, y \in \mathbb{C}$ when $a = x - b$ and $b = y$, then $|(x - y) + y| - |y| \leq |x - y|$ which implies

$$|x| - |y| \leq |x - y|$$

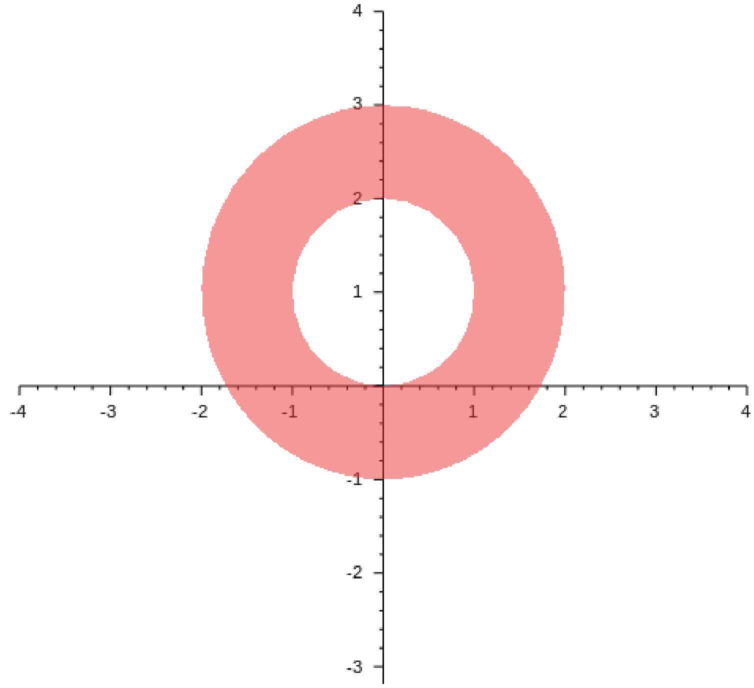
With this, we then have

$$|z - w| = |z(-v + v) - w| = |(z - v) - (w - v)| \geq |z - v| - |w - v| = |z - v| - |v - w|$$

taking advantage of the fact that $|x| = |-x|$ for all $x \in \mathbb{C}$ in the rightmost equality.

(b) Graph the points $z \in \mathbb{C}$ such that $1 < |z - i| < 2$

The following region is the set of points, note that the edges of the region are not included.



(c) For $z, w \in \mathbb{C}$ with $|z| < 1$ and $|w| = 1$, prove $|(w - z)/(1 - \overline{z}w)| = 1$

We have the following sequence of equations due to the fact that $|\overline{a}| = |a|$, $|ab| = |a||b|$, $|w| = 1$, and $\overline{a - b} = \overline{a} - \overline{b}$ for any $a, b \in \mathbb{C}$.

$$\begin{aligned}
\left|\frac{w - z}{1 - \overline{z}w}\right| &= \frac{|w - z|}{|1 - \overline{z}w|} \\
&= \frac{|w - z|}{||w| - \overline{z}w|} \\
&= \frac{|w - z|}{|\overline{w}w - \overline{z}w|} \\
&= \frac{|w - z|}{|w(\overline{w} - \overline{z})|} \\
&= \frac{|w - z|}{|w||\overline{w} - \overline{z}|} \\
&= \frac{|w - z|}{|\overline{w} - \overline{z}|} \\
&= \frac{|w - z|}{|\overline{w - z}|} \\
&= \frac{|w - z|}{|w - z|} \\
&= 1
\end{aligned}$$

7

Let $x, y, z \in \mathbb{R}$ and define

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}$$

(a) Prove the above $d(\cdot,\cdot)$ satisfies the triangle inequality

If $|x-z| \le |y-z|$ or $|x-z| \le |x-y|$, then the fact that $f(t) = t/(1+t)$ is an increasing function implies the triangle inequality for d.

So assume that $|x-z|$ is greater than both $|x-y|$ and $|y-z|$. Since $|x-z| \le |x-y| + |y-z|$ we have

$$\frac{|x-z|}{1+|x-z|} \le \frac{|x-y|}{1+|x-z|} + \frac{|y-z|}{1+|x-z|} \le \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$$

with the rightmost inequality comming from our initial assumption. Thus the triangle inequality holds for this $d(\cdot,\cdot)$.

(b)

The proof for this is identical to the proof of in the previous part of this problem after substituting in the function $|\cdot-\cdot|$ for $g(\cdot,\cdot)$. This is because the only property of $|\cdot-\cdot|$ that was used in the previous proof was that it upholds the triangle inequality, which $g(\cdot,\cdot)$ also does.

8

(a)

Let $f_1(x)/f_2(x), g_1(x)/g_2(x) \in \mathcal{R}$.
By the following, the zero constant is the additive identity:

$$f_1(x)/f_2(x) + 0 = f_1(x)/f_2(x) = 0 + f_1(x)/f_2(x)$$

By the following, the one constant is the multiplicative identity:

$$\left(f_1(x)/f_2(x)\right)(1) = f_1(x)/f_2(x) = (1)\left(f_1(x)/f_2(x)\right)$$

Then

$$f_1(x)/f_2(x) + g_1(x)/g_2(x) = \frac{f_1(x)g_2(x) + g_1(x)f_2(x)}{g_2(x)f_2(x)}$$

so $\mathcal{R}$ is closed under addition. Since polynomial addition is commutative and associative, then addition is commutative and associative in $\mathcal{R}$. Finally, since $f(x) + (-f(x)) = 0$, additive inverses exist in $\mathcal{R}$.
Since

$$\left(f_1(x)/f_2(x)\right)\left(g_1(x)/g_2(x)\right) = f_1(x)g_1(x)/f_2(x)g_2(x)$$

then $\mathcal{R}$ is closed under multiplication. Since polynomial multiplication is commutative and associative, then multiplication is commutative and associative in $\mathcal{R}$. Finally, if $f_1(x)/f_2(x)$ is not zero, then

$$\left(f_1(x)/f_2(x)\right)\left(f_2(x)/f_1(x)\right) = 1$$

and so multiplicative inverses exist in $\mathcal{R}$.

Finally, since polynomial addition and multiplication obides by the distributive law, then so does addition and multiplication in $\mathcal{R}$.

So $\mathcal{R}$ is indeed a field.

(b)

(c)