Math 508: Advanced Analysis
Homework 3

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1 Give an example of closed sets $V_j \subset \mathbb{R}$ where $\cup_{j=1}^{\infty} V_j$ is open.

Define $V_j = [-j, j]$. Then $\cup_{j=1}^{\infty} V_j = \mathbb{R}$ and $\mathbb{R}$ is open. But this example leaves a bad taste in one's mouth since the union of the $\{V_j\}$ is the whole space, $\mathbb{R}$. As for a more slick example, define $V_j = [-(1-1/j), 1-1/j]$. In this case $\cup_{j=1}^{\infty} V_J = (-1, 1)$, which is open in $\mathbb{R}$.

2 Construct a bounded set of real numbers with exactly three limit points

Define

$$V_x = \left\{ x + \frac{1}{n} \middle| n \in \mathbb{Z}^+ \right\}$$

With this definition, x is the only limit point of V_x and is bounded since it is contained within the ball $B_1(x)$. Therefore $V_1 \cup V_2 \cup V_3$ is a bounded set with exactly three limit points.

3 Find the interior points and boundary points of each set, and describe the sets closure

(a) $(0, 1] \subset \mathbb{R}$

Interior points: $(0, 1)$
Boundary points: $\{0, 1\}$
Closure: $[0, 1]$

(b) $\mathbb{R}^2 \subset \mathbb{R}^3$ (the coordinate plane $z = 0$)

Interior points: There are no interior points because there is no ball of any point completely contained in the plane.
Boundary points: The set itself is the set of boundary points.
Closure: This set is closed, so the closure is itself.

(c) $\mathbb{Q} \subset \mathbb{R}$

Interior points: There are no interior points. The set is countable, and therefore discrete, so no neighborhood of any point can be completely contained within $\mathbb{R}$.
Boundary points: $\mathbb{R}$
Closure: $\mathbb{R}$

(d)

The graph of the function

$$y = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

as a subset of $\mathbb{R}^2$. Call the set G.
Interior points: There are no interior points.
Boundary points: $G \cup \{(0, y) \mid |y| \leq 1\}$
Closure: Same as the boundard points.

4 Which sets are compact? Why?

(a) $[0, 1] \subset \mathbb{R}$

This set is a closed and bounded subset of $\mathbb{R}$, so it's compact by Heine-Borel.

(b) $\{0\} \cup \{1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\} \subset \mathbb{R}$

Again, this set is a closed and bounded subset of $\mathbb{R}$, so it's compact by Heine-Borel.

(c) X=$[0, 1] \setminus \mathbb{Q}$ as a subset of $\mathbb{R}$

Since X contains no rational numbers then $0 \notin X$, however, every open ball of zero will contain a point of X since it will contain an irrational in $[0, 1]$. Therefore 0 is a limit point of X, but since it's not contained in X, X is not closed. Because X is not closed as a subset of $\mathbb{R}$ then it is not compact, again by Heine-Borel.

5

For any element $x_k \in S$ we have that

$$\frac{1}{10} x_k = \frac{9}{10^2} + \frac{9}{10^3} + \frac{9}{10^4} + \cdots + \frac{9}{10^{k+1}}$$

so

$$\begin{aligned}
x_k - \frac{1}{10} x_k &= \frac{9}{10} - \frac{9}{10^{k+1}} \\
\left(1 - \frac{1}{10}\right) x_k &= \frac{9}{10} - \frac{9}{10^{k+1}} \\
(10 - 1) x_k &= 9 - \frac{9}{10^k} \\
9 x_k &= 9 - \frac{9}{10^k} \\
x_k &= 1 - \frac{1}{10^k}
\end{aligned}$$

which gives us a closed form for each $x_k \in S$. With this formula, it's obvious that 1 is an upper bound. Assume for later contradiction that there exists an $a < 1$ where a is an upper bound. However, for any integer $\overline{k} > -\log_{10}(1-a)$ we have

$$x_{\overline{k}} = 1 - \frac{1}{10^{\overline{k}}} > 1 - \frac{1}{10^{-\log_{10}(1-a)}} = 1 - 10^{\log_{10}(1-a)} = 1 - (1 - a) = a$$

which contradicts our assumption that a is an upper bound. Thus 1 is the $\sup S$.

6

Define $\langle A, B \rangle$ for $A, B \in \mathcal{M}_{k,n}$ by $\langle A, B \rangle = \text{trace}(AB^t)$

(a) Show this $\langle \cdot, \cdot \rangle$ is an inner product.

Positive Definite Let $A \in \mathcal{M}_{k\times n}$ with $r_1, r_2, \ldots, r_k$ being the k vectors that make up the rows of A. Then $\langle A, A\rangle = \text{trace}(AA^t) = r_1 r_1^t + r_2 r_2^t + \cdots + r_k r_k^t$. Since each $r_i r_i^t$ is simply the dot product of r_i with itself, then each $r_i r_i^t \geq 0$ with equality when r_i is the zero vector. Thus $\langle A, A\rangle \geq 0$ with equality when A is the zero matrix.

Additivity Let $A, B, C \in \mathcal{M}_{k\times n}$. By properties of the trace function and matrix multiplication we have

$$\langle A+B, C\rangle = \text{trace}\left((A+B)C^t\right) = \text{trace}\left(AC^t + BC^t\right) = \text{trace}(AC^t) + \text{trace}(BC^t) = \langle A, C\rangle + \langle B, C\rangle$$

Homogeneity Let $A, B \in \mathcal{M}_{k\times n}$ and $\alpha \in \mathbb{R}$. By properties of the trace function we have

$$\langle \alpha A, B\rangle = \text{trace}(\alpha AB^t) = \alpha\, \text{trace}(AB^t) = \alpha \langle A, B\rangle$$

Symmetry Let $A, B \in \mathcal{M}_{k\times n}$. By properties of the trace function we have

$$\langle A, B\rangle = \text{trace}\left(AB^t\right) = \text{trace}\left((AB^t)^t\right) = \text{trace}\left(BA^t\right) = \langle B, A\rangle$$

(b) Let $|A|^2 = \langle A, A\rangle$ and define $d(A, B) := |A - B|$. Show d is a metric.

Non-negativity

$$d(A, B) = |A - B| = \sqrt{\langle A, B\rangle} \geq \sqrt{0} = 0$$

Symmetry

$$d(A, B) = |A - B| = \sqrt{\langle A, B\rangle} = \sqrt{\langle B, A\rangle} = |B - A| = d(B, A)$$

Triangle Inequality

$$d(A, B) = |A - B| = |A - C + C - B| = |(A - C) + (C - B)| \leq |A - C| + |C - B| = d(A, C) + d(C, B)$$

(c)

(d)

7

Let K be compact in $\mathbb{R}^n$ with $x \notin K$. Further let $d = \inf\{d(x, y) \mid y \in K\}$. There must be a limit point $z \in \mathbb{R}^n$ of K such that $d(x, z) = d$, If there were not such a point, then we'd be able to find some $r > 0$ such that $B_r(p) \cap K = \{\}$ for p with $d(x, p) = d$. This would imply that, say, $d + \frac{1}{2}r$ would be a lower bound on the set $\{d(x, y) \mid y \in K\}$, which is a contradiction since d is the infimum of that set. Now because z is a limit point of K and K is compact, then it is also closed by Heine-Borel. Hence $z \in K$, as desired.

8

Let $X = \mathbb{R} - \{0\}$ with the normal metric on $\mathbb{R}$ of the absolute value of the difference between two points. Then the sets $A = [-1, 0)$ and $B = (0, 1]$ are closed sets since zero is not in X. They are obviously dijsoint. Furthermore $\text{dist}(A, B) = 0$ since each has points arbitrarily close to zero.

9

Let $\{x_n\}$ be a sequence of points in $\mathbb{R}^2$ that contains every point with rational coordinates. Let $\{r_n\}$ be a sequence of positive real numbers such that $\sum_n r_n = 1$. Define

$$U = \bigcup_n D(x_n, r_n)$$

where $D(x, r)$ is the disc of radius r centered at x.

(a) Show U is open and dense in $\mathbb{R}$

The set U is open because it is the union of open sets. Also, the set U is dense in $\mathbb{R}^2$ because $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and U contains $\mathbb{Q}^2$ as a subset.

(b) Show that no straight line L is completely contained in U

Let L be a line in the plane. A line is the linear combination of two specified points, so let those points be u and v. Then L is the set of points $(1-c)u + cv = u + c(v-u)$ for all $c \in \mathbb{R}$. By way of contradiction, assume that L is completely contained in U. Then for any $n \in \mathbb{N}$, we have that $D(x_n, r_n) \cap L = \{u + c(v-u) \mid c \in (s_n, t_n)\}$ for some $s_n, t_n \in \mathbb{R}$. Thus by defining the function $f : L \to \mathbb{R}$ by $f(u + c(v-u)) = c$ we see that $f(D(x_n, r_n) \cap L) = (s_n, t_n)$. So for all $n \in \mathbb{N}$, we get an open cover of the real line. For instance, we can cover the closed interval $[0, 3]$, but since $[0, 3]$ is compact, we can find a finite open subcover $(s_{n_1}, t_{n_1}), \ldots, (s_{n_k}, t_{n_k})$. However, this implies that

$$3 \leq \sum_{i=1}^{k} (t_{n_i} - s_{n_i}) \leq \sum_{n=1}^{\infty} (t_n - s_n) \leq \sum_{i=1}^{\infty} 2r_n = 2 \sum_{i=1}^{\infty} r_n = 2$$

and we thus arrive at a contradiction. Hence L is not completely contained in U.

10 Product Topology

Let (E_1, d_1) and (E_2, d_2) be two metric spaces and define the distance on $E_1 \times E_2$ to be

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$$

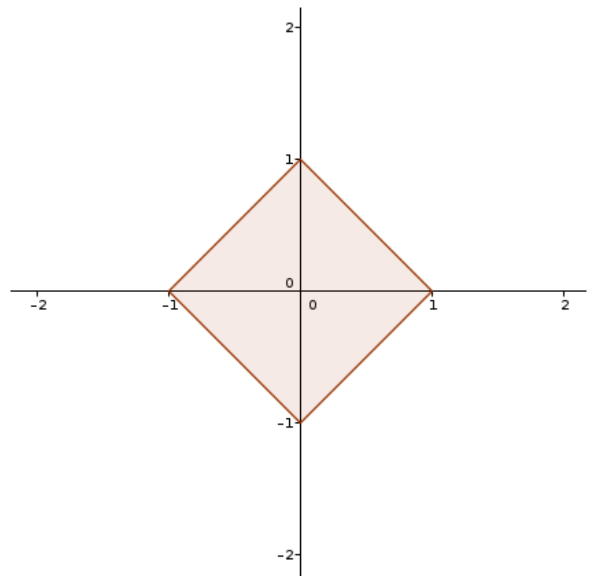
for each $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$.

(a) Warmup

Viewing $\mathbb{R}^2$ as $\mathbb{R} \times \mathbb{R}$ we have

$$d((0, 0), (1, 2)) = d(0, 1) + d(0, 2) = 1 + 2 = 3$$

and the "disc" centered at the origin is



(b)

Let $U \subset E_1 \times E_2$ be an open subset. Let $(u_1, u_2) \in U$. We can thus find an open ball $B_r(u_1, u_2) \subset U$. Define $U_1 = \{x \in E_1 \mid d((x, u_2), (u_1, u_2)) < r\}$. With this definition for any $x \in U_1$, the open ball of radius $\frac{r - d_1(x,u_1)}{2}$ will be contained in U_1 since any $y \in B_{\frac{r-d_1(x,u_1)}{2}}(x)$ has that

$$\begin{aligned}
d((y, u_2), (u_1, u_2)) &= d_1(y, u_1) + d_2(u_2, u_2) \\
&= d_1(y, u_1) \\
&\leq d_1(y, x) + d_1(x, u_1) \\
&< \frac{r - d_1(x, u_1)}{2} + d_1(x, u_1) \\
&= \frac{r - d_1(x, u_1) + 2d_1(x, u_1)}{2} \\
&= \frac{r + d_1(x, u_1)}{2} \\
&< \frac{r + r}{2} \\
&= r
\end{aligned}$$

so that $d((y, u_2), (u_1, u_2)) < r$. Thus U_1 is open in E_1. Similarly, we also have that the set $U_2 = \{x \in E_2 \mid d((u_1, x), (u_1, u_2)) < r\}$ is open in E_2 because each $x \in U_2$ has the open ball $B_{\frac{r-d_2(x,u_2)}{2}}(x)$ around it which is completely contained in U_2. Finally, with the simple definitions of U_1 and U_2, we see $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset B_r(u_1, u_2) \subset U$, as desired.

Conversely, assume that for each point $(u_1, u_2) \in U$ there exist U_1, U_2 open in E_1, E_2, respectively, with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$. So fix $(u_1, u_2) \in U$. As a consequence of the hypothesis at the opening of this paragraph, we can find open balls $B_{r_1}(u_1) \subset U_1$ and $B_{r_2}(u_2) \subset U_2$. Define $r = \min(r_1, r_2)$. Then for any point $(x, y) \in B_r((u_1, u_2))$ we have

$$d_1(u_1, x) \leq d_1(u_1, x) + d_2(u_2, y) = d((u_1, x), (u_2, y)) < r \leq r_1$$

and

$$d_2(u_2, y) \leq d_1(u_1, x) + d_2(u_2, y) = d((u_1, x), (u_2, y)) < r \leq r_2$$

which implies $x \in B_{r_1}(u_1)$ and $y \in B_{r_2}(u_2)$, respectively. In turn, this means $(x, y) \in U_1 \times U_2$ and thus (x, y) is also in U. Hence the open ball $B_r((u_1, u_2))$ is contained in U, i.e. U is open.

11 The p-adic topology on the rational numbers.

(a) Show that $(\mathbb{Q}, d_p)$ is a metric space.

Fix a prime p.

Lemma 11.1. *For $x, y, z \in \mathbb{Q}$, $d(x, z) \leq \max(d(x, y), d(y, z))$.*

Proof. Let x, y, z be rational numbers. Then we have

$$x - y = p^{\nu_1} \frac{n_1}{k_1}, \quad y - z = p^{\nu_2} \frac{n_2}{k_3} \quad \text{and} \quad x - z = p^{\nu_3} \frac{n_3}{k_2}$$

where p does not divide any n_i or k_i. With this in place we assume without loss of generality that $\nu_1 \leq \nu_2$ so that

$$x - z = (x - y) + (y - z) = p^{\nu_1} \frac{n_1}{k_1} + p^{\nu_2} \frac{n_2}{k_3} = p^{\nu_1} \left(\frac{n_1}{k_1} + p^{\nu_2 - \nu_1} \frac{n_2}{k_3} \right) = p^{\nu_1} \left(\frac{n_1 k_2 + p^{\nu_2 - \nu_1} n_2 k_1}{k_1 k_2} \right)$$

Now with both the above and below forms of $x - z$, we see $\nu_3 \geq \nu_1$, which given our assumption that $\nu_1 \leq \nu_2$, implies $\nu_3 \geq \min(\nu_1, \nu_2)$. Hence we have

$$d(x, z) = |x - z|_p = p^{-\nu_3} \leq p^{-\min(\nu_1, \nu_2)} = \max\left(p^{-\nu_1}, p^{-\nu_2}\right) = \max\left(|x - y|_p, |y - z|_p\right) = \max\left(d(x, y), d(y, z)\right)$$

as desired. $\square$

Non-negativity Let $x, y \in \mathbb{Q}$ be distinct. Then $d(x, y) = |x - y|_p = p^{-\nu}$ for some ν. Since p is positive, then $d(x, y)$ will always be positive. On the other hand $d(x, x) = |x - x|_p = 0$.

Symmetry Let $x, y \in \mathbb{Q}$ with $x - y = p^{\nu} \frac{n}{k}$. Then $y - x = p^{\nu} \frac{-n}{k}$, implying that $|x - y|_p = |y - x|_p = p^{-\nu}$. Hence $d(x, y) = d(y, x)$.

Triangle Inequality Let $x, y, z \in \mathbb{Q}$. Making use of the lemma above, we obtain

$$d(x - z) \leq \max\left(d(x, y), d(y, z)\right) \leq d(x, y) + d(y, z)$$

(b) For $x, a \in \mathbb{Q}$ show that $x \in N_r(a)$ implies $N_r(x) = N_r(a)$.

Let $x, y, a \in \mathbb{Q}$ with $r > 0$ and $x \in N_r(a)$. Then $d(x, a) < r$.

So for any $y \in N_r(x)$, $d(x, y) < r$. In particular, $d(y, a) \leq \max(d(x, a), d(x, y)) < \max(r, r) = r$, by the above lemma. So $y \in N_r(a)$, i.e. $N_r(x) \subset N_r(a)$.

On the other hand, we similarly have that for any $y \in N_r(a)$, $d(y, a) < r$ and therefore $d(x, y) \leq \max(d(x, a), d(a, y)) < \max(r, r) = r$ again by the above lemma. Thus $y \in N_r(x)$, meaning $N_r(a) \subset N_r(x)$.

Combining these two results leaves us with $N_r(x) = N_r(a)$.

(c) Show that any two neighborhoods are disjoint, or one is contained in the other.

Let $a, a' \in \mathbb{Q}$ and r, r' be positive reals. If $N_r(a)$ and $N_{r'}(a')$ are disjoint, we are done. So assume not. Then there is an $x \in N_r(a) \cap N_{r'}(a')$. By the previous part of the problem, we then have $N_r(x) = N_r(a)$ and $N_{r'}(x) = N_{r'}(a')$. Thus by assuming without loss of generality that $r < r'$ we obtain

$$N_r(a) = N_r(x) \subset N_r'(x) = N_r'(a')$$

as desired.