Math 508: Advanced Analysis
Homework 5

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1 Determine the divergence or convergence of $\sum \frac{1}{1+z^n}$ for complex z

If $|z| \leq 1$ the $|z^n| \leq 1$ which implies $|1+z^n| \leq 2$ and subsequently that

$$\left|\frac{1}{1+z^n}\right| \geq \frac{1}{2}$$

Therefore $\frac{1}{1+z^n} \not\to 0$, implying the divergence of $\sum \frac{1}{1+z^n}$ in this case.

Now assume that $|z| > 1$. Then $|z^n|$ increases as n gets large and $|z^n + 1| > 2$. So let N be such that $|z^n| > 2$ for all $n > N$. Then

$$|z^n + 1| \geq |z^n| > |z^n| - 1 \geq |z^n| - \frac{|z^n|}{2} = \frac{|z|^n}{2} \tag{1.1}$$

for all $n > N$. By separating $\sum \frac{1}{|1+z^n|}$ like so

$$\sum_{n=1}^{\infty} \frac{1}{|1+z^n|} = \sum_{n=1}^{N} \frac{1}{|1+z^n|} + \sum_{n=N+1}^{\infty} \frac{1}{|1+z^n|}$$

equation 1.1 informs us that

$$\sum_{n=1}^{\infty} \frac{1}{|1+z^n|} < \sum_{n=1}^{N} \frac{1}{|1+z^n|} + \sum_{n=N+1}^{\infty} \frac{2}{|z|^n} = \sum_{n=1}^{N} \frac{1}{|1+z^n|} + \sum_{n=N+1}^{\infty} \left(\frac{\sqrt[n]{2}}{|z|}\right)^n$$

Now the left addend of the right-hand side of the above inequality is a finite sum and the right addend is a convergent geometric series since $0 < \frac{\sqrt[n]{2}}{|z|} < 1$. Thus the right-hand side of the above inequality converges, implying the convergence of the left-hand side, $\sum_{n=1}^{\infty} \frac{1}{|1+z^n|}$. This in turn implies that our series in question is absolutely convergent when $|z| > 1$, and therefore convergent.

2 If $a_n > 0$, $\sum a_n$ converges, and $\{b_n\}$ is bounded show that $\sum a_n b_n$ converges.

Let $\sum a_n$ be a convergent series with each $a_n > 0$ and $\{b_n\}$ a bounded sequence. Define M as the bound on $\{b_n\}$ so that $|b_n| < M$ for all n. Then we have

$$\sum |a_n b_n| = \sum a_n |b_n| < \sum a_n M = M \sum a_n$$

so due to the convergence of $\sum a_n$, then $M \sum a_n$ and therefore $\sum |a_n b_n|$ converges. Hence $\sum a_n b_n$ converges absolutely, and so converges.

3 Find the radius of convergence of the following power series.

(a) $\sum n^3 z^n$

Since

$$\left|\frac{(n+1)^3 z^{n+1}}{n^3 z^3}\right| = \left|\frac{n^3 + 3n^2 + 3n + 1}{n^3} z\right|$$

approaches z as n increases, then the ratio test tells us that the radius of convergence is 1.

(b) $\sum \frac{2^n}{n!} z^n$

This series is a power series of $e^2 z$:

$$\sum \frac{2^n}{n!} z^n = \sum \frac{2z^n}{n!} = e^{2z}$$

and so this series converges for all z, i.e. the radius is infinite.

(c) $\sum n! z^n$

Since

$$\left| \frac{(n+1)! z^{n+1}}{n! z^3} \right| = |(n+1)z|$$

approaches ∞ as n increases, then the ratio test tells us that the radius of convergence is 0, but technically, $z = 0$ will make the series converge.

4 Detemine the limit of $\{a_n\}$ where $a_0 = 0$ and $a_{n+1} = a_n^2 + \frac{4}{25}$ for $n \geq 1$

We first note that if the limit is finite, say L, then it must satisfy $L^2 - L + \frac{4}{25} = 0$ given the recursive definition. Therefore, L can only be $\frac{1}{5}$ or $\frac{4}{5}$ since these are the roots of that equation.

Now since $a_0 = 0$, the recusive formula just adds a positive value of $\frac{4}{25}$, informing us that this sequence if monotonically increasing. Furthermore, if $a_n < \frac{1}{5}$ we see that

$$a_{n+1} = a_n^2 + \frac{4}{25} < \left(\frac{1}{5}\right)^2 + \frac{4}{25} = \frac{1}{5}$$

which simultaneously implies that the sequence is bounded and rules out $4/5$ as a possible limit. Thus the sequence is monotonically increasing and bounded, implying that it must converge, and because the initial value is zero, the only point it can converge to is $\frac{1}{5}$.

5 If $a_n \geq 0$ and $\sum a_n$ converges, show that $\sum \frac{\sqrt{a_n}}{n}$ converges

Let $a_n \geq 0$ and assume $\sum a_n$ converges. Then the partial sums $\{s_n\}$ with $s_n = a_1 + \cdots + a_n$ converge. This implies the convergence of $\{t_n\}$ where $t_n = \sqrt{a_1} + \cdots + \sqrt{a_n}$ since each $a_n \geq 0$. Thus defining $\sum x_n = \sum \sqrt{a_n}$ implies that $\{t_n\}$ is the sequence of partial sums of $\sum x_n$ and they form a bounded sequence. Furthermore, defining $\sum y_n$ as the harmonic series, we have $y_0 \geq y_1 \geq y_2 \geq \cdots$ and $\lim_{n\to\infty} y_n = 0$. Thus Rudin's Theorem 3.42 informs us that the series

$$\sum x_n y_n = \sum \frac{\sqrt{a_n}}{n}$$

converges.

6 Find N so that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > 100$

Define $\sum a_n$ to be the harmonic series. We have that

$$\begin{aligned}
\sum a_n &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} \\
&= 1 + \frac{1}{2} \sum_{k=0}^{\infty} 2^k \frac{1}{2^k} \\
&= 1 + \frac{1}{2} \sum_{k=0}^{\infty} 2^k a_{2^k}
\end{aligned}$$

Because we know $\sum a_n$ diverges, we also have that $\sum_{k=0}^{\infty} 2^k a_{2^k}$ diverges. So if we can find an integer K such that $1 + \frac{1}{2} \sum_{k=0}^{K} 2^k a_{2^k} > 100$ then for $N = 2^K$ we'll have $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N} > 100$. Thus since

$$1 + \frac{1}{2} \sum_{k=0}^{K} 2^k a_{2^k} = 1 + \frac{1}{2} \sum_{k=0}^{K} 2^k \frac{1}{2^k} = 1 + \frac{K}{2}$$

then $1 + \frac{K}{2} > 100$ implies that $K > 198$, so $K = 199$ and therefore $N = 2^{199}$ will satisfy.

7 Determine whether or not $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \cdots$ converges

By grouping the series like so

$$\left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) - \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots$$

we see that it is no different than $\sum c_n$ where

$$c_n = (-1)^{n+1} \left(\frac{1}{2n-1} + \frac{1}{2n}\right)$$

This is an alternating series for which $\left(\frac{1}{2n-1} + \frac{1}{2n}\right)$ converges to zero, so our original series must also converge by Rudin's theorem 3.43.

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(a)

(b)

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Let A be an $n \times n$ matrix, and use the norm of homoework 3 problem 6.

(a) Compute $(I - A)(I + A + A^2 + \cdots + A^N)$

$$(I - A)(I + A + A^2 + \cdots + A^N) = I - A^{N+1}$$

(b) Show that if $|A| < 1$, then $I - A$ is invertible.

Let $|A| < 1$. Then as N gets large, $I - A^{N+1}$ approaches I, so in light of the previous part of this problem, if $(I - A)\sum_{n=0}^{N} A^n$ converges as N gets large, then $(I - A)\left(\sum A^n\right)$ will be I and therefore $I - A$ will be invertibile. Since

$$\lim_{N\to\infty}(I - A)\sum_{n=0}^{N} A^n = (I - A)\lim_{N\to\infty}\sum_{n=0}^{N} A^n = (I - A)\sum_{n=0}^{\infty} A^n$$

we'll have that $I - A$ is invertible if $\sum_{n=0}^{\infty} A^n$ converges. Thus because $\sum |A^n| \leq \sum |A|^n$, the right side converges since it's a geometric series and $|A| < 1$. This tells us that $\sum A^n$ converges absolutely and that therefore $\sum A^n$ converges. Hence we have our desired result of $I - A$ being invertible.

(c) Show the set of invertible matrices is open

Let $\varepsilon = \frac{1}{|A^{-1}|}$ and B be a matrix in the ε-ball of A. Then we have that $|A - B| < \varepsilon$ implying that $|A^{-1}||A - B| < 1$. Thus $|I - A^{-1}B| < 1$. By the previous part of the problem, we then know that $I - \left(I - A^{-1}B\right) = A^{-1}B$ is invertible. Then there is some invertible matrix C with $CA^{-1}B = I$. But then B is invertible with inverse CA^{-1}. Hence the ε-ball of A contains only invertible elements, and so the set of invertible matrices is open.

11 Prove that the set of othogonal $n \times n$ matrices are compact.

Since $n \times n$ matrices are a subspace of $\mathbb{R}^{n^2}$, then to show this set is compact, we need only show that it's closed and bounded.

Bounded. Since any othogonal matrix A has that $AA^t = I$, then with our definition of the norm, $|A|^2 = \langle A, A\rangle = \operatorname{trace}(AA^t) = \operatorname{trace}(I) = n$. Thus the set of othogonal matrices is bounded.

Closed.

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(a)

(b)