Math 508: Advanced Analysis
Homework 6

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1 Prove $\cos x$ and $\sin x$ are continuous for all $x \in \mathbb{R}$

$\cos x$ is continuous: Let $x \in \mathbb{R}$ For any $|h|$ we have

$$\begin{aligned} |\cos(x+h) - \cos x| &= |\cos x \cos h - \sin h \sin x - \cos x| \\ &= |\cos x(\cos h - 1) + \sin h(-\sin x)| \\ &\leq |\cos x(\cos h - 1)| + |\sin h(-\sin x)| \\ &\leq |\cos h - 1| + |\sin h| \end{aligned}$$

where the last inequality comes from the fact that $-1 \leq -\sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all values of x. Now, as $h \to 0$, $\cos h \to 1$ and $\sin h \to 0$. This implies the last line in the equation above approaches zero as $h \to 0$. Thus, for any $\varepsilon > 0$ we can find a small enough δ so that for $|h| < \delta$ $|\sin(x+h) - \sin x| < \varepsilon$. Hence $\sin x$ is continuous at x.

$\sin x$ is continuous: Let $x \in \mathbb{R}$ For any $|h|$ we have

$$\begin{aligned} |\sin(x+h) - \sin x| &= |\sin x \cos h + \sin h \cos x - \sin x| \\ &= |\sin x(\cos h - 1) + \sin h \cos x| \\ &\leq |\sin x(\cos h - 1)| + |\sin h \cos x| \\ &\leq |\cos h - 1| + |\sin h| \end{aligned}$$

where the last inequality comes from the fact that $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$ for all values of x. Like we saw in the previous paragraph, the last line of the above equation can be made arbitrarily close to zero as $h \to 0$. Hence, similar to what we saw in the previous paragraph, $\cos x$ is continuous at x.

2

Let $f(x) = x^2 + 4x$ and $0 < \varepsilon < 4$. Define $\delta = \frac{\varepsilon}{5}$. Then $0 < \delta < 1$, given the constraints on ε. Assume that $|x| < \delta$, so that $0 < |x| < \delta < 1$, in particular $0 < |x|^2 < |x| < 1$. With this and the triangle inequality, we then have

$$|f(x)| = |x^2 + 4x| \leq |x^2| + |4x| = |x|^2 + 4|x| < |x| + 4|x| = 5|x| < 5\delta = \varepsilon$$

as desired.

3 Prove the existence of an $x \in [1,2]$ such that $x^5 + 2x + 5 = x^5 + 10$

Put $f(x) = x^5 + 2x + 5$ and $g(x) = x^5 + 10$. Being polynomials, f and g are both continuous. Therefore the intermediate value theorem implies that on the interval $[1,2]$ f will take on every value between $f(1) = 8$ and $f(2) = 41$ and g will take on every value between $g(1) = 11$ and $g(2) = 26$. Since $f(1) = 8 < 11 = g(1)$ and $g(2) = 26 < 41 = f(2)$, f and g must therefore intersect on the interval $[1,2]$. Hence there's a point x on $[1,2]$ with $f(x) = g(x)$, i.e. where $x^5 + 2x + 5 = x^5 + 10$.

4 Show there exists two diametrically opposite points on the Eath's equator that have the same temperature.

We'll use the 3-dimensional cartesian coordinate system and orient the earth so that the z-axis contains the earth's axis of rotation, the north pole has a positive z coordinate, and the center of the earth is at the orgin. Then the equator will be contained in the xy-plane. Finally, we will denote, by p_θ for $\theta \in \mathbb{R}_{\geq 0}$, the point on the equator that is intersected by the line through the origin and $(\cos\theta, \sin\theta, 0)$.

Now let $T : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the function whose value at $\theta \in \mathbb{R}_{\geq 0}$ is the temperature of p_θ. We will assume that this temperature function is continuous. Define the function $D : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by $D(\theta) = T(\theta) - T(\theta + \pi)$, i.e. it is the difference in temperature of p_θ and its diametrically opposite point, $p_{\theta+\pi}$. Since D is made up of the composition and difference of continuous functions, then it too is continuous.

Let p_θ be a point on the equator that doesn't have the same temperature as its diametric opposite (there must be such a point, otherwise there would be nothing to prove as all points diametrically opposite would have the same temperature). Then $D(\theta) = -D(\theta + \pi)$ and neither values are zero. Without loss of generality, assume that $D(\theta) < 0 < D(\theta + \pi)$. Since D is continuous and it's domain, $\mathbb{R}_{\geq 0}$, is connected, then this and the intermediate value theorem implies that there is a $\theta' \in (\theta, \theta + \pi)$ such that $D(\theta') = 0$. But this implies that $T(\theta') = T(\theta' + \pi)$, i.e. the diametrically opposed points $p_{\theta'}$ and $p_{\theta'+\pi}$ have the same temperature.

5

Define $\{a_n\}$ and $\{b_n\}$ by $a_n = -1/n$ and $b_n = 1/n$. Then of course both sequences converge to zero. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x} & \text{otherwise} \end{cases}$$

has the desired property of $f(a_n) \to 0$ and $f(b_n)$ being unbounded.

Does there exist such a function that's continuous at $x = 0$ This is not possible as it would contradict Rudin's Theorem 4.2.

6

Let $f(a, n) = (1 + a)^n$ where a and n are positive.

(a) Behavior of $f(a, n)$ for constant a or n

For constant a how does $f(a, n)$ behave as $n \to \infty$? When a is constant, $1 + a > 1$ since a was assumed positive. So $f(a, n) = (1 + a)^n \to \infty$ as $n \to \infty$.

For constant n how does $f(a, n)$ behave as $a \to 0$? As $a \to 0$, $1 + a \to 1$. So for constant n, $f(a, n) = (1 + a)^n \to 1$ as $a \to 0$.

(b)

Defining $a_n = \sqrt[n]{L + \frac{1}{n}} - 1$, then a_n is a sequence of positive values, and furthermore

$$f(a_n, n) = (1 + a_n)^n = \left(1 + \left(\sqrt[n]{L + \frac{1}{n}} - 1\right)\right)^n = \left(\sqrt[n]{L + \frac{1}{n}}\right)^n = L + \frac{1}{n}$$

so that $f(a_n, n) \to L$ as $n \to \infty$.

7 Which of the following functions are uniformly continuous on $[0, \infty)$

(a) $f(x) = x \sin x$

For any $x \in [0, \infty)$ and $h \in \mathbb{R}$ we have that

$$\begin{aligned}
|(x+h)\sin(x+h) - x\sin x| &= |h\sin(x+h) + x\sin(x+h) - x\sin x| \\
&= |h\sin(x+h) + x(\sin x\cos h + \sin h\cos x) - x\sin x| \\
&= |h\sin(x+h) + x(\sin x\cos h + \sin h\cos x - \sin x)|
\end{aligned}$$

For a fixed h, $h\sin(x+h)$ is bounded, but $x(\sin x\cos h + \sin h\cos x - \sin x)$ gets arbitrarily large as x gets large. Thus no matter how small h is, we can find x large enough so that $|h\sin(x+h) + x(\sin x\cos h + \sin h\cos x - \sin x)|$, and therefore $|(x+h)\sin(x+h) - x\sin x|$, can be made arbitrarily large. Hence there is no way for $f(x) = x\sin x$ to be uniformly continuous on $[0, \infty)$.

(b) $f(x) = e^x$

For any $x \in [0, \infty)$ and $h \in \mathbb{R}$ we have that

$$|e^{x+h} - e^x| = |e^x(e^h - 1)|$$

so no matter the value of h, $|e^x(e^h - 1)|$ can be made arbitrarily large, and therefore so can $|e^{x+h} - e^x|$. Hence e^x cannot be uniformly continuous on $[0, \infty)$

(c) $f(x) = \frac{1}{1+x}$

For any $\varepsilon > 0$, set $\delta = \varepsilon$. Thus for any $x, y \in [0, \infty$ when $|y - x| < \delta$ we have

$$\left|\frac{1}{1+y} - \frac{1}{1+x}\right| = \left|\frac{(1+y)-(1+x)}{(1+y)(1+x)}\right| = \frac{|y-x|}{|(1+y)(1+x)|}$$

Since $x, y \in [0, \infty)$ then $1 + y \geq 1$ and $1 + x \geq 1$ which implies

$$\left|\frac{1}{1+y} - \frac{1}{1+x}\right| = \frac{|y-x|}{|(1+y)(1+x)|} \leq |y - x| < \delta = \varepsilon$$

so that $f(x)$ is uniformly continuous on $[0, \infty)$

8 Show that $f(x) = \sqrt{x}$ is continuous $\forall x \geq 0$

Let $\varepsilon > 0$ and $x \in [0, \infty)$. Define $\delta = \varepsilon\sqrt{x}$ so that in particular $\varepsilon = \frac{\delta}{\sqrt{x}}$. Then for any $y \geq 0$ such that $|x - y| < \delta$ we have

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x} + \sqrt{y}|} \leq \frac{|x-y|}{\sqrt{x}} < \frac{\delta}{\sqrt{x}} = \varepsilon$$

indicating that f is continuous at x.

Is the function uniformly continuous? Yes. First note that because f is continous, as per above, then because $[0, 1]$ is compact in $\mathbb{R}$ f is uniformly continuous on $[0, 1]$. Furthermore, for any $x, y \in [1, \infty)$, $\sqrt{y} + \sqrt{x} \geq 1$. Therefore for any $\varepsilon > 0$, setting $\delta = \varepsilon$ means that when $|y - x| < \delta$ we have

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{|\sqrt{x} + \sqrt{y}|} \leq |x - y| < \delta = \varepsilon$$

so that f is uniformly continuous on $[1, \infty)$.

Finally for any $\varepsilon > 0$, define $\delta = \min(\delta^-, \delta^+)$ where δ^- is such that $|x - y| < \delta^- \Rightarrow |\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}$ for all $x, y \in [0, 1]$ and δ^+ is such that $|x - y| < \delta^+ \Rightarrow |\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}$ for all $x, y \in [1, \infty)$. Thus with this definition, whether $x, y \in [0, 1]$ or $x, y \in [1, \infty)$, $|x - y| < \delta$ will imply $|\sqrt{x} - \sqrt{y}| < \epsilon$.

It remains to be seen that $|x - y| < \delta$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$ when $x \in [0, 1]$ and $y \in [1, \infty)$. This nevertheless holds, since for $x \in [0, 1]$ and $y \in [1, \infty)$, $|x - y| < \delta$ implies both $|x - 1| < \delta < \delta^-$ and $|y - 1| < \delta < \delta^+$. These two equations in turn imply $|\sqrt{x} - \sqrt{1}| < \frac{\varepsilon}{2}$ and $|\sqrt{y} - \sqrt{1}| < \frac{\varepsilon}{2}$. Hence we obtain

$$|\sqrt{x} - \sqrt{y}| = |\sqrt{x} - 1| + |\sqrt{y} - 1| = |\sqrt{x} - \sqrt{1}| + |\sqrt{y} - \sqrt{1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore f is uniformly continous on all of $[0, \infty)$.

9

(a) Prove $[0, 1]$ and $\mathbb{R}$ are not homeomorphic.

Since $[0, 1]$ is compact but $\mathbb{R}$ is not, there exists no continuous function from $[0, 1]$ to $\mathbb{R}$. Thus there exists no homeomorphism between them either.

(b) Prove $\mathbb{R}$ and $(0, \infty)$ are homeomorphic.

The function $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$ is both bijective and continuous, and thus is a homeomorphism between the two sets.

(c) Prove $\mathbb{R}^2$ and $\{(x, y) \in \mathbb{R}^2 : y > 0\}$ are homeomorphic.

Since $f(x) = e^x$ is a homeomorphism from $\mathbb{R} \to (0, \infty)$ then the function $g : \mathbb{R}^2 \to \{(x, y) \in \mathbb{R}^2 : y > 0\}$ defined by

$$g((x, y)) = (x, e^y)$$

will also be a homeomorphism.

(d) Prove $\mathbb{R}$ and $(-1, 1)$ are homeomorphic.

The map $f : (-1, 1) \to \mathbb{R}$ is continuous since its composed of the quotient, product, and difference of continuous functions. It's injective, because

$$\begin{aligned} \frac{x}{x^2-1} &= \frac{y}{y^2-1} \\ x(y^2-1) &= y(x^2-1) \\ xy^2 - x &= yx^2 - y \\ xy^2 + y &= yx^2 + x \\ y(xy+1) &= x(yx+1) \\ y &= x \end{aligned}$$

It's surjective because for any $y \in \mathbb{R}$, if it were to be true that

$$y = \frac{x}{x^2-1}$$

then we'd have

$$yx^2 - x - y = 0$$

and the quadratic formula yields

$$x = \frac{1 + \sqrt{1+4y^2}}{2y}$$

as one of the roots. Thus because

$$\frac{1 + \sqrt{1+4y^2}}{2y} < \frac{\sqrt{1+4y^2}}{2y} < \frac{\sqrt{4y^2}}{2y} = \frac{2y}{2y} = 1$$

then we have a solution for $x < 1$. Thus f is a homeomorphism.

10

Let

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that f is continuous on $\mathbb{R}$

The function f is the composition and product of functions x, $\sin x$, and $1/x$, which are all continuous at nonzero reals. Therefore f is continuous at all nonzero reals. It remains to be shown that f is continuous at zero.

Let $\varepsilon > 0$ and set $\delta = \varepsilon$. The for $|x| < \delta$ we have

$$|x\sin(1/x) - f(0)| = |x\sin(1/x) - 0| = |x\sin(1/x)| \leq |x| < \delta = \varepsilon$$

so that f is continuous at zero.

(b) Is f uniformly continous on $[0, 2\pi]$?

Yes, f is continous on $[0, 2\pi]$ and $[0, 2\pi]$ is compact.

(c)

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(a)

(b)

12 Show a continuous real-valued function f that has $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ must be $f(x) = cx$ for some c.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and have that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Put $c = f(1)$. Then for integer n, $f(n) = f(1) + f(n-1) = f(1) + \cdots + f(1) = nf(1) = nc$. Thus for any rational n/m we have

$$mf(n/m) = f(n/m) + \cdots + f(n/m) = f(m(n/m)) = f(n) = nc$$

so that $f(n/m) = (n/m)c$. Thus any rational $q \in \mathbb{Q}$ has $f(q) = qc$. With this, we finally have that for $x \in \mathbb{R}$ and a rational sequence $\{a_n\}$ with $a_n \to r$, the continuity of f yields

$$f(r) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n = cr$$

as desired.

13

Let $E \subset \mathbb{R}$ be a set and $f : E \to \mathbb{R}$ be uniformly continuous.

(a) Show that if E is bounded, then so is $f(E)$

Lemma 13.1. *If $E \subset X$ is a bounded set, then so is its closure, $\overline{E}$.*

Proof. Let E be a bounded subset of a metric space X, and x, y two points of its closure, $\overline{E}$. If $x, y \in E$, then there's nothing to prove since E is already bounded. We must address two remaining case:

1. when one of x and y is a limit point not in E and the other a point of E
2. when both x and y are limit points not in E

Assume the first, and without loss of generality, let $x \in E$ and y be a limit point of E. Then any open ball centered at y, say $B_1(y)$, contains some point of $\overline{x} \in E$. Thus $d(x, y) \leq d(x, \overline{x}) + d(\overline{x}, y) < M + 1$ where M is a bound for the set E. Hence the distance between any point of E and a limit point is bounded.

Now assume the second case above. By the previous paragraph we have $d(x, y) \leq d(x, z) + d(z, y) < 2(M + 1)$ for any $z \in E$. So in this case, the distance is bounded.

Hence we have that the closure of E is bounded. □

Let $E \subset \mathbb{R}$ be a bounded set. Then the closure, $\overline{E}$, is also bounded by the above lemma. Hence it's closed and bounded, implying that it's compact as a subset of $\mathbb{R}$. Therefore, the set $f(\overline{E})$ is also compact since f is continuous. Thus $f(\overline{E})$ is bounded since it's a subset of $\mathbb{R}$. But because $E \subset \overline{E}$, then $f(E) \subset f(\overline{E})$, and so $f(E)$ must also be bounded.

(b) If E is not bounded, give an example showing $f(E)$ might not be bounded.

Let $E = (0, \infty)$ and $f(x) = x$. Then f is uniformly continuous, E is not bounded, and $f(E) = (0, \infty)$ which is also not bounded.

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