Math 508: Advanced Analysis
Homework 8

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1 Show that $\cos x$ is differentiable at all x.

For $h > 0$ we have

$$\begin{aligned}
\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} &= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= (\cos x)(0) - (\sin x)(1) \\
&= -\sin x
\end{aligned}$$

where we make use of the trigonometric identity $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and the fact that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

and

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

2 Derivatives of matrices

Let $A(t)$ be an $n \times n$ matrix whoose elements depend smoothly on t and that is invertible at t_0.

(a) Compute the derivative of $A^2(t)$ in terms of A and A'

The derivative of $A^2(t)$ is the derivative with respect to t of each of its elements as functions of t. So

$$\frac{d}{dt}\left(A^2(t)\right)_{ij} = \frac{d}{dt}\sum_{k=1} n a_{ik}a_{kj} = \sum_{k=1} n\left(a'_{ik}a_{kj} + a_{ik}a'_{kj}\right) = (A'A + AA')_{ij}$$

so that we have $\frac{d}{dt}A^2(t) = A'A + AA'$. Note the multiplication of A and A' may not commute.

(b) Show $A(t)$ is invertible for all t near t_0

We know that a matrix is invertible if and only if it has nonzero determinent. Since the determinent of A is simply the sum and product of entries of A and the entries of A depend smoothly on t, then $\det(A(t))$ also depends smoothly on t. Since $A(t_0)$ is invertible, then $\det(A(t_0)) \neq 0$, which, along with the previous sentence, implies that there is a neighborhood of t_0 for which all t in that neighborhood have $\det(A(t)) \neq 0$.

(c) Find a formula for the derivative of $A^{-1}(t)$ at t_0

For $h > 0$ we have.

$$\begin{aligned}
\lim_{h\to 0} \frac{A^{-1}(t_0+h) - A^{-1}(t_0)}{h} &= \lim_{h\to 0} \frac{\left(A^{-1}(t_0)A(t_0)\right)A^{-1}(t_0+h) - A^{-1}(t_0)\left(A(t_0+h)A^{-1}(t_0+h)\right)}{h} \\
&= A^{-1}(t_0) \lim_{h\to 0} \frac{A(t_0) - A(t_0+h)}{h} A^{-1}(t_0+h) \\
&= A^{-1}(t_0)\left(-\lim_{h\to 0} \frac{A(t_0+h) - A(t_0)}{h} A^{-1}(t_0+h)\right) \\
&= A^{-1}(t_0)\left(-A'(t_0)\right)\left(\lim_{h\to 0} A^{-1}(t_0+h)\right) \\
&= -A^{-1}(t_0)A'(t_0)A^{-1}(t_0)
\end{aligned}$$

so that $\left(A^{-1}\right)' = -A^{-1}A'A^{-1}$

(d) Find a formula for the derivative of $A^{-2}(t)$

By the first part of this problem we have

$$\frac{d}{dt}\left(A^{-2}\right) = \left(A^{-1}\right)' A^{-1} + A^{-1}\left(A^{-1}\right)'$$

which, through use of the previous part of the problem, leads to

$$\frac{d}{dt}\left(A^{-2}\right) = \left(-A^{-1}A'A^{-1}\right) A^{-1} + A^{-1}\left(-A^{-1}A'A^{-1}\right) = -\left(A^{-1}A'A^{-2} + A^{-2}A'A^{-1}\right)$$

3

(a) Find the unique solution $v(x)$ such that $v' = v$ and $v(0) = c$ for constant c.

Let $v(x)$ be such that $v' = v$ and $v(0) = c$. Define $u(x) = \frac{1}{c}v(x)$. Then

$$u'(x) = \frac{1}{c}v'(x) = \frac{1}{c}v(x) = u(x) \tag{3.1}$$

and

$$u(0) = \frac{1}{c}v(0) = \frac{1}{c}c = 1 \tag{3.2}$$

Thus, $u(x)$ must be e^x since e^x is the unique function with the properties in equations 3.1 and 3.2. But then we have $e^x = u(x) = \frac{1}{c}v(x)$ implying $v(x) = ce^x$.

(b) Prove that $e^{x+a} = e^a e^x$ for all real a and x

Define $u(x) = e^{x+a} - e^a e^x$ for real a. Then we have

$$u(0) = e^a - e^a = 0$$

and

$$u'(x) = e^{x+a} - e^a e^x = u(x)$$

by the previous part of this problem, $u(x) = 0e^x = 0$. Hence $e^{x+a} = e^a e^x$.

(c) For constant γ show $v' - \gamma v \leq 0$ implies $v(x) \leq v(0)e^{\gamma x}$ for $x \geq 0$

Let $v' - \gamma v \leq 0$ and define $g(x) = e^{-\gamma x}v(x)$. Then we have

$$g' = -\gamma e^{-\gamma x}v + v'e^{-\gamma x} = e^{-\gamma x}\left(v' - \gamma v\right) \leq 0$$

which implies that g is always decreasing so that in particular $g(x) \leq g(0)$ for all $x \geq 0$. That is to say that

$$\begin{aligned} g(x) &\leq g(0) \\ e^{-\gamma x}v(x) &\leq e^{-\gamma 0}v(0) \\ e^{-\gamma x}v(x) &\leq v(0) \\ v(x) &\leq v(0)e^{\gamma x} \end{aligned}$$

for all $x \geq 0$.

4 Show that a continuous $f : [a, b] \to \mathbb{R}$ can be approximated by a piecewise linear function $g : [a, b] \to \mathbb{R}$

Let $\varepsilon > 0$ be given. Because f is continuous and $[a, b]$ is compact, then f is uniformly continuous, and we can thus find a $\delta > 0$ such that

$$|x - y| < \delta \;\;\Rightarrow\;\; |f(x) - f(y)| < \varepsilon \tag{4.3}$$

We divide $[a, b]$ up into chuncks of size less than δ. Let n be an integer such that $n\delta > b$ so that $\delta > \frac{b}{n}$. Hence, each of $[a, a+\overline{\delta}), [a+\overline{\delta}, a+2\overline{\delta}), \ldots, [a+(n-1)\overline{\delta}, b)$, where $\overline{\delta} = \frac{b-a}{n}$, are intervals of size less than δ. Define $M_i = \sup f(x)$ and $m_i = \inf f(x)$ for each $i \in \{0, 1, \cdots, n-1\}$ and $x \in [a + i\overline{\delta}, a + (i+1)\overline{\delta})$ so that in particular

$$\left| f(x) - \frac{M_i - m_i}{2} \right| < \varepsilon \tag{4.4}$$

for any $x \in [a + i\overline{\delta}, a + (i+1)\overline{\delta})$ by equation 4.3. Further define a set of functions $\varphi_i : [a, b] \to \{0, 1\}$ by

$$\varphi_i(x) = \begin{cases} 1 & x \in [a + i\overline{\delta}, a + (i+1)\overline{\delta}) \\ 0 & \text{otherwise} \end{cases}$$

so that we may define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = \delta_b(x) f(b) + \sum_{i=0}^{n-1} \varphi_i(x) \frac{M_i - m_i}{2}$$

where δ_b is the Kronecker function. Thus for any $x \in [a + i\overline{\delta}, a + (i+1)\overline{\delta})$ we have

$$|f(x) - g(x)| = \left| f(x) - \frac{M_i - m_i}{2} \right| < \varepsilon$$

by equation 4.4 and for $x = b$ we have $|f(x) - g(x)| = |f(b) - f(b)| = 0 < \varepsilon$ as desired.

5

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function.

(a) If $f'(1) = f''(1) = f'''(1) = 0$ and $f''''(1) > 0$, show f has a local minimum at $x = 1$.

Let $f'(1) = f''(1) = f'''(1) = 0$ and $f''''(1) > 0$. Then there is an $\varepsilon > 0$ such that $f''''(x) > 0$ for $x \in (1 - \varepsilon, 1 + \varepsilon)$. So if $1 < x < 1 + \varepsilon$ then repeated applications of The Fundamental Theorem of Calculus tells us

$$\begin{aligned}
f(x) - f(1) &= \int_1^x f'(y)\; dy \\
&= \int_1^x f'(y) - f'(1)\; dy \\
&= \int_1^x \int_1^y f''(z)\; dzdy \\
&= \int_1^x \int_1^y f''(z) - f''(1)\; dzdy \\
&= \int_1^x \int_1^y \int_1^z f'''(w)\; dwdzdy \\
&= \int_1^x \int_1^y \int_1^z f'''(w) - f'''(1)\; dwdzdy \\
&= \int_1^x \int_1^y \int_1^z \int_1^w f''''(s)\; dsdwdzdy \\
&> 0
\end{aligned}$$

which implies $f(x) > f(1)$. Furthermore, for any x with $1-\varepsilon < x < 1$ we have again by repeated applications of The Fundamental theorem of Calculus

$$\begin{aligned}
f(x) - f(1) &= -\int_x^1 f'(y)\ dy \\
&= -\int_x^1 f'(y) - f'(1)\ dy \\
&= \int_x^1 \int_y^1 f''(z)\ dzdy \\
&= \int_x^1 \int_y^1 f''(z) - f''(1)\ dzdy \\
&= -\int_x^1 \int_y^1 \int_z^1 f'''(w)\ dwdzdy \\
&= -\int_x^1 \int_y^1 \int_z^1 f'''(w) - f'''(1)\ dwdzdy \\
&= \int_x^1 \int_y^1 \int_z^1 \int_w^1 f''''(s)\ dsdwdzdy \\
&> 0
\end{aligned}$$

again so that $f(x) > f(1)$. Hence if $x \in (1-\varepsilon, 1+\varepsilon)$ then $f(x) \geq f(1)$ so that $f(1)$ is a local minimum.

(b) What can be said about f near $x = 1$ when $f'(1) = f''(1) = 0$ and $f'''(1) > 0$.

There is nothing that can be generally said about f since it may not be a local maximum or minimum at all.

6

Let $u(x)$ be a smooth solution to the differential equation

$$u'' + 3u' - (1+x^2)u = 0$$

(a) Show that u cannot have a positive local maximum

Given the differential equation above, we have

$$u'' + 3u' = (1+x^2)u$$

for $u(x)$ so that at any local maximum x_0 where $u(x_0) = 0$, we have

$$u'' = (1+x_0^2)u \tag{6.5}$$

So if $u(x_0)$ is positive then $u''(x_0) > 0$, i.e. u is convex at x_0. Thus if $u(x_0)$ is positive then u can only have a local minimum at x_0.

(b) Show that u cannot have a negative local minimum

Let x_0 be as in the previous part of this problem. Equation 6.5 also implies that if $u(x_0)$ is negative, then so is $u''(x_0)$. Hence u can only have a negative local maximum at x_0.

(c) If $u(0) = u(2) = 0$, show that $u(x) = 0$ for $x \in [0, 2]$

If u is zero on $[0, 2]$, then we are done. So let $x_0 \in (0, 2)$ be nonzero. Either

1. $u(x_0) > 0$ or
2. $u(x_0) < 0$

If the first case, then since u is smooth, it is bounded on $[0, 2]$. Thus there must be a maximum positive value of u on $[0, 2]$. But this contradicts the first part of this problem. This case can thus not happen.

If the second case, then since u is smooth it is bounded on $[0, 2]$. Thus there must be a minimum negative value of u on $[0, 2]$. But this contradicts the second part of this problem. Thus this case can also not happen.

Hence, the only possible scenario is $u(x) = 0$ for $x \in [0, 2]$.

(d)

7

(a)

(b)

8 Use the Reimann sum to compute $\int_0^b \sin x \; dx$

Define a partition P_n of $[0, b]$ by $P = \{0, \theta, 2\theta, \dots, (n-1)\theta, b\}$ where $\theta = b/n$. Then we have

$$U(P_n, \sin x) = \theta(\sin\theta + \cdots + \sin(n\theta))$$

and

$$L(P_n, \sin x) = \theta(\sin 0 + \sin\theta + \cdots + \sin((n-1)\theta))$$

so that $U(P_n, \sin x) - L(P_n, \sin x) = \theta \sin(n\theta)$ which approaches zero as $n \to \infty$. Thus we know that $\sin x$ is indeed integrable.

Now to find the actual value of the integral of $\sin x$ we evaluate $\lim_{n\to\infty} U(P_n, \sin x)$. So because $\theta = b/n$ we have the following.

$$\begin{aligned}
\lim_{n\to\infty} U(P_n, \sin x) &= \lim_{n\to\infty} \theta \sum_{k=1}^{n} \sin(k\theta) \\
&= \lim_{n\to\infty} \theta \left(\frac{\cos(\theta/2) - \cos((n+1/2)\theta)}{2\sin(\theta/2)} \right) \\
&= \left(\lim_{n\to\infty} \cos(\theta/2) - \cos((n+1/2)\theta) \right) \left(\lim_{n\to\infty} \frac{\theta}{2\sin(\theta/2)} \right) \\
&= (\cos(0) - \cos(b))\,(1) \\
&= 1 - \cos(b)
\end{aligned}$$

so that $\int_0^b \sin x \; dx = 1 - \cos(b)$

9 Define $f(0) = 3$ and $f(x) = \sin(1/x)$ for $x \in (0, 2/\pi]$. Show f is Reimann integrable.

Let $\varepsilon > 0$ be given. Then on the interval $[\varepsilon/8, 2\pi]$ f is continuous and therefore Reimann integrable on the interval. Hence there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon/2$. Define a partition $P' = P \cup \{0\}$. Then we have $U(P',f) = U(P,f) + 3\frac{\varepsilon}{8}$ and $L(P',f) = L(P,f) - \varepsilon/8$. This implies

$$U(P',f) - L(P',f) = U(P,f) + 3\frac{\varepsilon}{8} - L(P,f) + \varepsilon/8 = \varepsilon/2 + \Big(U(P,f) - L(P,f)\Big) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus f is Riemann integrable.

10

11 Prove the Integral Mean Value Theorem

Let f be real and continuous on $[a,b]$. Partition $[a,b]$ by $P = \{a,b\}$. Then $L(P,f) = (b-a)f(y)$ and $U(P,f) = (b-a)f(z)$ where $f(y) = \max(f(x))$ and $f(z) = min(f(x))$ for $x \in [a,b]$. Thus because

$$L(P,f) \leq \int_a^b f(x)\ dx \leq U(P,f)$$

we have

$$f(y) \leq \frac{1}{b-a}\int_a^b f(x)\ dx \leq f(z)$$

Since f is continuous on $[y,z]$ then The Intermediate Value Theorem tells us that there is a $c \in [y,x]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\ dx = f(c)$$

as desired.