Math 508: Advanced Analysis
Homework 9

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1

Let $c \in \mathbb{R}$ be positive, $f \in \mathscr{R}$ be an even function, and $g \in \mathscr{R}$ and be an odd function.

(a) Show $\int_{-c}^{c} f(x)dx = 2\int_0^c f(x)dx$

We can separate $\int_{-c}^{c} f(x)dx$ as

$$\int_{-c}^{c} f(x)dx = \int_{-c}^{0} f(x)dx + \int_{0}^{c} f(x)dx$$

Through a change of variable, setting $x = h(u)$ where $h(u) = -u$, we obtain

$$\begin{aligned}
\int_{-c}^{c} f(x)dx &= \int_{c}^{0} f(h(u))h'(u)du + \int_{0}^{c} f(x)dx \\
&= \int_{c}^{0} f(-u)(-1)du + \int_{0}^{c} f(x)dx \\
&= -\int_{c}^{0} f(u)du + \int_{0}^{c} f(x)dx \\
&= \int_{0}^{c} f(u)du + \int_{0}^{c} f(x)dx \\
&= \int_{0}^{c} f(x)dx + \int_{0}^{c} f(x)dx \\
&= 2\int_{0}^{c} f(x)dx
\end{aligned}$$

since f is even.

(b) Show $\int_{-c}^{c} g(x)dx = 0$

We can separate $\int_{-c}^{c} g(x)dx$ as

$$\int_{-c}^{c} g(x)dx = \int_{-c}^{0} g(x)dx + \int_{0}^{c} f(x)dx$$

Through a change of variable, setting $x = h(u)$ where $h(u) = -u$, we obtain

$$\begin{aligned}
\int_{-c}^{c} g(x)dx &= \int_{c}^{0} g(h(u))h'(u)du + \int_{0}^{c} g(x)dx \\
&= \int_{c}^{0} g(-u)(-1)du + \int_{0}^{c} g(x)dx \\
&= -\int_{c}^{0} -g(u)du + \int_{0}^{c} g(x)dx \\
&= \int_{c}^{0} g(u)du + \int_{0}^{c} g(x)dx \\
&= -\int_{0}^{c} g(u)du + \int_{0}^{c} g(x)dx \\
&= -\int_{0}^{c} g(x)dx + \int_{0}^{c} g(x)dx \\
&= 0
\end{aligned}$$

since g is odd.

(c) Show $\int_{-c}^{c} f(x)g(x)dx = 0$

Since $f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x)$ then $f(x)g(x)$ is an odd function. By the previous part of this problem

$$\int_{-c}^{c} f(x)g(x)dx = 0$$

2 Find function f and constant c so that $\int_0^x f(t)e^{3t}dt = c + x - \cos(x^2)$

Let's guess that $f(t) = e^{-3t}(1 + 2t\sin(t^2))$ so that

$$\int_0^x f(t)e^{3t}dt = \int_0^x (1 + 2t\sin(t^2))dt = \int_0^x dt + \int_0^x 2t\sin(t^2)dt = x + \int_0^x 2t\sin(t^2)dt \tag{2.1}$$

Defining $g(s) = \sqrt{s}$ we have $g(0) = 0$ and $g(x^2) = x$ so that by putting $t = g(s)$ we have

$$\begin{aligned}
\int_0^x 2t\sin(t^2)dt &= \int_0^{x^2} 2(g(s))\sin((g(s))^2)g'(s)ds \\
&= \int_0^{x^2} 2(\sqrt{s})\sin(s)\left(\frac{1}{2\sqrt{s}}\right)ds \\
&= \int_0^{x^2} \sin(s)ds \\
&= -\cos(x^2) + \cos(0) \\
&= 1 - \cos(x^2)
\end{aligned}$$

through a change of variable. Substituting this result back into equation 2.1 we obtain

$$\int_0^x f(t)e^{3t}dt = 1 + x - \cos(x^2)$$

so that we see our definition of f works with $c = 1$.

3

Let $f(x) = \sqrt{9 + x^4}$ and define a partition P of $[0, 2]$ by $P = \{0 = x_0, x_1, \ldots, x_N = 2\}$ where $x_j - x_{j-1} = \Delta x = 2/N$. Since f is a monotonically increasing function on $[0, 2]$, then

$$U(P, f) = \sum_{i=1}^{N} f(x_i)\Delta x = \frac{2}{N}\sum_{i=1}^{N} f(x_i)$$

and

$$L(P, f) = \sum_{i=1}^{N} f(x_{i-1})\Delta x = \frac{2}{N}\sum_{i=1}^{N} f(x_{i-1})$$

leaving us with an error in estimation of

$$U(P, f) - L(P, f) = \frac{2}{N}\sum_{i=1}^{N} f(x_i) - \frac{2}{N}\sum_{i=1}^{N} f(x_{i-1}) = \frac{2}{N}(f(2) - f(0)) = \frac{2}{N}(5 - 3) = \frac{4}{N}$$

Hence, if we'd like the error in our computation of $\int_0^2 \sqrt{9 + x^4}dx$ to be less than $1/100$, then we need $4/N < 1/100$, i.e. $N > 400$.

4

Let $f(s)$ be a smooth function and c be a constant. Define

$$u(x,t) = \int_{x-ct}^{x+ct} f(s)ds$$

Applying the fundamental theorem of calculus then yields the following four equations

$$\begin{array}{ll} \partial_x u = f(x+ct) - f(x-ct) & \partial_t u = cf(x+ct) + cf(x-ct) \\ \partial_x^2 u = f'(x+ct) - f'(x-ct) & \partial_t^2 u = c^2 f'(x+ct) - c^2 f'(x-ct) \end{array}$$

and so we see that

$$\partial_t^2 u = c^2 f'(x+ct) - c^2 f'(x-ct) = c^2 \left(f'(x+ct) - f'(x-ct)\right) = c^2 \partial_x^2 u$$

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(a)

If we set

$$u = \frac{1}{c}\left(-\cos(cx)\int_0^x f(t)\sin(ct)dt + \sin(cx)\int_0^x f(t)\cos(ct)dt\right) \tag{5.2}$$

then

$$u' = \sin(cx)\int_0^x f(t)\sin(ct)dt + \cos(cx)\int_0^x f(t)\cos(ct)dt$$

and

$$\begin{aligned} u'' &= c\left(\cos(cx)\int_0^x f(t)\sin(t)dt - \sin(cx)\int_0^x f(t)\cos(ct)dt\right) + \sin^2(cx)f(x) + \cos^2(cx)f(x) \\ &= c\left(\cos(cx)\int_0^x f(t)\sin(t)dt - \sin(cx)\int_0^x f(t)\cos(ct)dt\right) + f(x)\left(\sin^2(cx) + \cos^2(cx)\right) \end{aligned}$$

Hence $u'' + c^2 u = f(x)$.

(b)

Let $0 < c < 1$. Let $u(x)$ and $w(x)$ be solutions to the differential equation $w'' + c^2 w = f(x)$ and be zero on the boundary points of $[0, \pi]$. Then define $v(x) = w(x) - u(x)$. We would then have

$$v'' + c^2 v = w'' - u'' + c^2(w+u) = (w'' + c^2 w) - (u'' + c^2 u) = f(x) - f(x) = 0 \tag{5.3}$$

Thus the v must have the form

$$v = a\sin(cx) + b\cos(cx)$$

for some a and b as it is the solution of the homogenous equation in 5.3. Hence $v(0) = a\sin(0) + b\cos(0) = b = w(0) - u(0) = 0$ so that b is 0. Furthermore $v(\pi) = a\sin(c\pi) = w(\pi) - u(\pi) = 0$ so that $a\sin(c\pi) = 0$. Since $0 < c < 1$, then $a = 0$. Thus $v(x) = 0$, which implies w and u are the same. Hence there is only one unique solution when $0 < c < 1$.

(c)

Let $c = 1$, in which case we have $u'' + u = f$. Thus with two applications of integration by parts we get

$$\begin{aligned}
\int_0^\pi f(x)\sin x\ dx &= \int_0^\pi u''\sin x\ dx + \int_0^\pi u\sin x\ dx \\
&= u'(\pi)\sin\pi - u'(0)\sin 0 - \int_0^\pi u' cos dx + \int_0^\pi u\sin x\ dx \\
&= -\int_0^\pi u' cos dx + \int_0^\pi u\sin x\ dx \\
&= -\left(u(\pi)\sin\pi - u(0)\sin 0 - \int_0^\pi u(-\sin x)dx\right) + \int_0^\pi u\sin x\ dx \\
&= -\left(-\int_0^\pi u(-\sin x)dx\right) + \int_0^\pi u\sin x\ dx \\
&= 0
\end{aligned}$$

(d)

Applying are result from part (a) to the situation when $c = 1$, the previous part implies equation 5.2 becomes

$$\begin{aligned}
u &= \frac{1}{c}\left(-\cos(cx)\int_0^x f(t)\sin(ct)dt + \sin(cx)\int_0^x f(t)\cos(ct)dt\right) \\
&= -\cos(x)\int_0^x f(t)\sin(t)dt + \sin(x)\int_0^x f(t)\cos(t)dt \\
&= \sin(x)\int_0^x f(t)\cos(t)dt
\end{aligned}$$

which is the unique solution for u.

6

Let L be the differential operator defined by $Lw = -w'' + c(x)w$ on the interval $J = [a, b]$, where $c(x)$ is some continuous function. Define the inner product by

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

(a) Show $\langle Lu, v\rangle = \langle u, Lv\rangle$ for u and v that are both zero on the boundary of J.

We first layout a helpful equation obtained via two applications of integration by parts:

$$\begin{aligned}
\int_a^b u''vdx &= u'(b)v(b) - u'(a)v(a) - \int_a^b u'v'dx \\
&= u'(b)(0) - u'(a)(0) - \int_a^b u'v'dx \\
&= -\int_a^b u'v'dx \\
&= -\left(u(b)v'(b) - u(a)v'(a) - \int_a^b uv''dx\right) \\
&= -\left((0)v'(b) - (0)v'(a) - \int_a^b uv''dx\right) \\
&= \int_a^b uv''dx
\end{aligned}$$

taking into account that the boundary points for u and v are zero, i.e. $u(a) = v(a) = u(b) = v(b) = 0$. With the above equation, our job is simple:

$$\begin{aligned}
\langle Lu, v\rangle &= \int_a^b (Lu)vdx \\
&= \int_a^b (-u'' + c(x)u)vdx \\
&= -\int_a^b u''vdx + \int_a^b c(x)uvdx \\
&= -\int_a^b uv''dx + \int_a^b c(x)uvdx \\
&= \int_a^b u(-v'')dx + \int_a^b u\left(c(x)v\right)dx \\
&= \int_a^b u(-v'' + c(x)v)dx \\
&= \int_a^b u(Lv)dx \\
&= \langle u, Lv\rangle
\end{aligned}$$

(b) If $Lu = \lambda_1 u$ and $Lv = \lambda_2 v$ for $\lambda_1 \neq \lambda_2$ then $\langle u, v\rangle = 0$

In considering $(\lambda_1 - \lambda_2)\langle u, v\rangle$, we use the previous part of this problem:

$$\begin{aligned}
(\lambda_1 - \lambda_2)\langle u, v\rangle &= \lambda_1\langle u, v\rangle - \lambda_2\langle u, v\rangle \\
&= \langle \lambda_1 u, v\rangle - \langle u, \lambda_2 v\rangle \\
&= \langle Lu, v\rangle - \langle u, Lv\rangle \\
&= \langle u, Lv\rangle - \langle u, Lv\rangle \\
&= 0
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ we must therefore have $\langle u, v\rangle$.

(c) If $Lu = 0$ and $Lv = f(x)$, show $\langle u, f\rangle = 0$

If $Lu = 0$ and $Lv = f$, then we have the following

$$\langle u, f\rangle = \langle u, Lv\rangle = \langle Lu, v\rangle = \langle 0, v\rangle = 0$$

7 Compute the arclength of $X(t) = (\cos t, \sin t, t)$ in $\mathbb{R}^3$

We have that

$$|X'(t)| = |(-\sin t, \cos t, 1)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

so that

$$\int_0^{4\pi} |X'(t)|dt = \int_0^{4\pi} \sqrt{2}dt = \sqrt{2}(4\pi - 0) = 4\pi\sqrt{2}$$

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(a)

This is virtually problem 5 of the previous homework.

(b)

The value $\|f\|_1$ has the following three properties

1. $\int_0^1 |f(x)|dx \geq 0$ with equality only when $f = 0$.
2. $\int_0^1 |cf(x)|dx = |c| \int_0^1 |f(x)|dx = |c|\|f\|$
3. $\int_0^1 |f(x) + g(x)|dx \leq \int_0^1 |f(x)| + |g(x)|dx = \|f\| + \|g\|$

which make it a norm on $C([0, 1])$.

(c)

10 Compute $\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|dx$

Since the graph of $|\sin(\lambda x)|$ for an arbitrary λ is just a sequence of concave "humps", we can find the area under a single "hump" and then multiply that times the fraction of these "humps" that are between 0 and 1. One such "hump" is the left-most one in $[0, 1]$. It's area is that of the area under $|\sin(\lambda x)|$ on the interval $[0, \frac{\pi}{\lambda}]$. Furthermore there are $\frac{\lambda}{\pi}$ of these "humps" over the interval $[0, 1]$. Hence we have

$$\lim_{\lambda\to\infty} \int_0^1 |\sin(\lambda x)|dx = \lim_{\lambda\to\infty} \frac{\lambda}{\pi} \int_0^{\frac{\pi}{\lambda}} |\sin(\lambda x)|dx = \frac{1}{\pi} \lim_{\lambda\to\infty} \lambda \int_0^{\frac{\pi}{\lambda}} |\sin(\lambda x)|dx$$

However, on the interval $[0, \frac{\pi}{\lambda}]$, $\sin(\lambda x)$ is positive and so from the above equation we get

$$\lim_{\lambda\to\infty}\int_0^1 |\sin(\lambda x)|dx = \frac{1}{\pi}\lim_{\lambda\to\infty}\lambda\int_0^{\frac{\pi}{\lambda}}\sin(\lambda x)dx$$

Now if we set $x = g(u)$ where $g(u) = \frac{u}{\lambda}$, by a change of variable, we get

$$\lim_{\lambda\to\infty}\int_0^1 |\sin(\lambda x)|dx = \frac{1}{\pi}\lim_{\lambda\to\infty}\lambda\int_0^{\pi}\sin\left(\lambda\left(\frac{u}{\lambda}\right)\right)\left(\frac{1}{\lambda}\right)du = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^{\pi}\sin(u)du$$

since $g'(u) = \frac{1}{\lambda}$, $g(\pi) = \frac{\pi}{\lambda}$, and $g(0) = 0$. Hence we are left with

$$\lim_{\lambda\to\infty}\int_0^1 |\sin(\lambda x)|dx = \frac{1}{\pi}\lim_{\lambda\to\infty}\int_0^{\pi}\sin(u)du = \frac{1}{\pi}\lim_{\lambda\to\infty}(-\cos(\pi)+\cos(0)) = \frac{1}{\pi}\lim_{\lambda\to\infty}0 = 0$$

11

Define $g(x) = f(x) - c$. Then $\lim_{x\to\infty} = 0$ and furthermore that

$$\frac{1}{T}\int_0^T f(x)dx = c + \frac{1}{T}\int_0^T g(x)dx$$

Without loss of generality we may assume that $c = 0$. Then, since f is coninuous and $\lim_{x\to\infty} = 0$, f is bounded. Thus there is an M such that $|f(x)| < M$. Let $\varepsilon > 0$ be given. Also becuase f is continuous, there exists a t such that for x with $0 < t < x$ we have $|f(x) - 0| < \varepsilon/2$. Hence we have the following sequence of equations

$$\begin{aligned}\left|\frac{1}{T}\int_0^T f(x)dx\right| &\le \frac{1}{T}\left(\int_0^t |f(x)|dx + \int_t^T |f(x)|dx\right)\\ &\le \frac{1}{T}\left(\int_0^t Mdx + \int_t^T \varepsilon/2dx\right)\\ &= \frac{Mt}{T} + \frac{(T-t)\varepsilon/2}{T}\end{aligned}$$

Thus for T such that $\frac{Mt}{T} < \varepsilon/2$, the right-hand side of the above equation becomes

$$\varepsilon/2 + \varepsilon/2\frac{T-t}{T} < \varepsilon$$

which implies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x)dx = 0$$

as desired.

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(a)

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)g(x) = 0$$

for all continuous functions $g(x)$. In particular if $g(x) = f(x)$ we have

$$\int_0^1 (f(x))^2 = 0$$

Since $f(x)^2 \ge 0$ we must then have $f(x)^2 = 0$, and so therefore $f(x) = 0$.

(b)

This is not true. By problem 6(b) of homework 6, the function

$$h(x) = \begin{cases} 0 & x \leq 0 \\ \left(\sqrt[x]{L + \frac{1}{x}}\right)^x & \text{otherwise} \end{cases}$$

approaches L as x approaches ∞. Hence we have that the function $g_{a,b}(x)$ defined by $h(x-a)h(b-x)$ is zero everywhere except for (a, b) where $a, b > 0$. Furthermore $g(x) \in C^1$

So, assume for later contradiction that $f(x) \neq 0$ is such that $\int_0^1 f(x)g(x) = 0$ for all $g(x)$ in C^1. Then there must be a point x_0 where f is positive. Then there exists $a, b \in [0, 1]$ with $a < b$ such that $x_0 \in (a, b)$ and f is positive on all of (a, b). However, since this is the case, then $fg_{a,b}$ where $g_{a,b}$ is as defined above will be positive on (a, b) and zero everywhere else, i.e.

$$\int_0^1 f(x)g_{a,b}(x)dx = 0$$

a contradiction.