Math 508: Advanced Analysis
Homework 10

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1

Let $L : S \to T$ be a linear map from vector spaces S to T. Let V_1 and V_2 be distinct solutions of the equation $LX = Y_1$. Futhermore let W be a solution to the equation $LX = Y_2$.

(a) Find a solution to $LX = 2Y_1 - 7Y_2$

Put $V = 2V_1 - 7W$. Then

$$LV = 2LV_1 - 7LW = 2Y_1 - 7Y_2$$

so that V is a solution to $LX = 2Y_1 - 7Y_2$.

(b) Find a solution (other than W) to $LX = Y_2$

Let V be as above and set $\overline{V} = V + 6W - 2V_1 + 6W$ so that

$$L\overline{V} = LV - 2LV_1 + 6LW = 2Y_1 - 7Y_2 - 2LY_1 + 6Y_2 = Y_2$$

Hence $\overline{V}$ is another solution to $LX = Y_2$.

2

Let $f(x) \in C([a, b])$ and put

$$\overline{f}(x) = \frac{1}{b-a}\int_a^b f(x)dx$$

Because $e^u \geq 1 + u$ for all u, then in particular we have $e^{f-\overline{f}} \geq 1 + f - \overline{f}$. Thus we have the following sequence

$$\begin{aligned}
\frac{e^{f(x)}}{e^{\frac{1}{b-a}\int_a^b f(x)dx}} &\geq 1 + f(x) - \frac{1}{b-a}\int_a^b f(x)dx \\
\int_a^b \frac{e^{f(y)}}{e^{\frac{1}{b-a}\int_a^b f(x)dx}}dy &\geq \int_a^b \left(1 + f(y) - \frac{1}{b-a}\int_a^b f(x)dx\right)dy \\
\frac{\int_a^b e^{f(y)}dy}{e^{\frac{1}{b-a}\int_a^b f(x)dx}} &\geq \int_a^b dy + \int_a^b f(y)dy - \int_a^b \left(\frac{1}{b-a}\int_a^b f(x)dx\right)dy \\
\frac{\int_a^b e^{f(y)}dy}{e^{\frac{1}{b-a}\int_a^b f(x)dx}} &\geq (b-a) + \int_a^b f(y)dy - \frac{b-a}{b-a}\int_a^b f(x)dx \\
\frac{\int_a^b e^{f(y)}dy}{e^{\frac{1}{b-a}\int_a^b f(x)dx}}dy &\geq b - a \\
\frac{1}{b-a}\int_a^b e^{f(y)}dy &\geq e^{\frac{1}{b-a}\int_a^b f(x)dx}
\end{aligned}$$

giving us the desired result.

3

4 Determine which of the following are pointwise and uniformly convergent

Since uniform convergence implies pointwise convergence, the proofs below omit any mention of pointwise convergence if the given sequence is proven to be uniformly convergent.

(a) $f_n(x) = \frac{\sin x}{n}$ on $\mathbb{R}$

Let $\varepsilon > 0$ be given. Choose integer N so that $N\varepsilon > 1$. Since $0 \leq |\sin x| \leq 1$ for all x, then

$$0 \leq \left|\frac{\sin x}{n}\right| \leq \frac{1}{n} < \varepsilon$$

for all $n \geq N$. Hence $f_n \to 0$ uniformly.

(b) $f_n(x) = \frac{1}{1+nx}$ on $[0, 1]$

This sequence converges pointwise to

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

since for $x = 0$ $f_n(x) = 1$ for all n. And furthermore, for any $\varepsilon > 0$, at a fixed $x_0 \in (0, 1]$ we can choose integer N such that $N > \frac{1}{x_0}\left(\frac{1}{\varepsilon} - 1\right)$ so that

$$\left|\frac{1}{1 + nx_0}\right| < \varepsilon$$

for all $n \geq N$

This sequence of functions, however, does not converge uniformly. Because each f_n is continuous on $[0, 1]$, converging uniformly would imply that f above is continuous, but it's not, due to the simple discontinuity at $x = 0$.

(c) $f_n(x) = \frac{x}{1+nx^2}$ on $\mathbb{R}$

For each n,

$$f_n'(x) = \frac{1}{1 + nx^2} - \frac{2nx^2}{(1 + nx^2)^2}$$

which is zero only at $x = \pm\sqrt{1/n}$. Since $f_n(x)$ is positive on $(0, \infty)$ and negative on $(-\infty, 0)$, we have that f_n has a minimum at $-\sqrt{1/n}$ and a maximum at $\sqrt{1/n}$. Because the minimum and maximum for each n are $f_n(-\sqrt{1/n}) = -1/2\sqrt{1/n}$ and $f_n(\sqrt{1/n}) = 1/2\sqrt{1/n}$, respectively, we have

$$0 \leq |f_n(x)| \leq \frac{1}{2}\sqrt{\frac{1}{n}} < 1/n$$

for all $n \in \mathbb{N}$. Hence, given any $\varepsilon > 0$, choosing integer N so that $N\varepsilon > 1$ gives us $|f_n(x)| < \varepsilon$ for all $n \geq N$ and all $x \in \mathbb{R}$. Thus $f_n \to 0$ uniformly.

5

Let $\{f_n\}$ and $\{g_n\}$ be sequences of functions in $C([0, 1])$. $f_n \to f$ and $g_n \to g$.

(a) If both $f_n \to f$ and $g_n \to g$ pointwise, does $f_n g_n \to fg$ pointwise fg?

Yes. For a fixed $x_0 \in [0, 1]$, $\{f_n(x_0)\}$ and $\{g_n(x_0)\}$ are just normal sequences, and so $f_n(x_0)g_n(x_0) \to f(x_0)g(x_0)$ since $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$.

(b) If both $f_n \to f$ and $g_n \to g$ uniformly, does $f_n g_n \to fg$ uniformly fg?

Let $\varepsilon > 0$ be given. Because all $\{f_n\}$ and $\{g_n\}$ are continuous, then their uniform convergence implies that both f and g are also continuous. Hence f and g are bounded on $[0, 1]$. Due to this, we can find an M such that $|f(x)| \leq M$ and $|g(x)| \leq M$. Furthermore, the uniform convergence of $f_n \to f$ and $g_n \to g$ we can find integer N such that both

$$|f_n(x) - f(x)| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3M}\right)$$

and

$$|g_n(x) - g(x)| < \min\left(\sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3M}\right)$$

for all $x \in [0, 1]$ and $n \geq N$. Hence we have

$$\begin{aligned}
|f_n g_n(x) - fg(x)| &= |(f_n(x) - f(x))(g_n(x) - g(x)) + f(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\
&\leq |f_n(x) - f(x)||g_n(x) - g(x)| + |f(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\
&< \left(\sqrt{\frac{\varepsilon}{3}}\right)^2 + M\frac{\varepsilon}{3M} + M\frac{\varepsilon}{3M} \\
&= \varepsilon
\end{aligned}$$

for all $n \geq N$ and all $x \in [0, 1]$, which implies that $f_n g_n \to fg$ uniformly.

6

This problem is identical to a problem on the previous homework.

7 Explain which conditions of the Contracting Map Theorem fail for the following

(a) $x \mapsto x + \frac{1}{x}$ on $[0, \infty)$

Denote this map by f. This map is not a contraction map because there is no $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \tag{7.1}$$

for all $x, y \in [0, \infty)$. This is because the difference between $d(x, y)$ and $d(f(x), f(y))$ is

$$\left|(x - y) - \left(x + \frac{1}{x} - y + \frac{1}{y}\right)\right| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{x - y}{xy}\right|$$

which can be made arbitrarily close to zero by making x and y large enough. This is problematic because the contraction condition in 7.1 implies that

$$(1 - \alpha)d(x, y) \leq d(f(x), f(y)) - d(x, y)$$

must be true; which is not the case when the difference on the righthand side above can be made arbitrarily close to zero.

(b) $x \mapsto \frac{x}{2}$ on $(0, 1]$

Denote this mapping by f. The failure here is that the metric space $(0, 1]$ is not complete. In particular, the method of successive approximations fails for this mapping since the sequence $\{x_n\}$ defined by $x_n = f(x_{n-1})$ for fixed, arbitrary $x_0 \in (0, 1]$ converges to zero, which is not in $(0, 1]$.

8

Let

$$f(x) = \sum_{k=0}^{\infty} \frac{\sin kx}{1+k^4}$$

(a) For which real x is f continuous?

Define $\{f_n\}$ to be the partial sums of the summation f. Since $|\sin(kx)| \leq 1$ for all k and all x and $1+k^4$ is increasing, then this sum converges for all x and therefore $f_n \to f$. Moreover, this means each f_n has a bound M_n and $\sum M_n$ converges. This then implies that $f_n \to f$ uniformly. Hence, because each f_n is continuous, f must also be continuous.

(b) Is f differentiable? Why?

Since $f(x)$ is a summation that converges for all x, then $f(x) < \infty$, and thus $f(x)$ is continuous since it is the composition of continuous functions. Furthermore, because

$$\int \sum_{i=1}^{\infty} \frac{k\cos(kx)}{1+k^4} = \sum_{i=1}^{\infty} \frac{k}{1+k^4} \int \cos(kx) = \sum_{i=1}^{\infty} \frac{k}{1+k^4} \frac{1}{k} \cos(kx) = \sum_{i=1}^{\infty} \frac{\cos(kx)}{1+k^4} = f(x)$$

then $f'(x) = \frac{k\cos(kx)}{1+k^4}$ by the Fundamental Theorem of Calculus.

9

For any complex number $z = x + iy$ and integer n we have $n^z = n^{x+iy} = n^x n^{iy}$. Furthermore we have

$$\left|n^{iy}\right| = \left|e^{\ln(n^{iy})}\right| = \left|e^{iy\ln(n)}\right| = 1$$

Thus for any sequence bounded by M and all $z \in \mathbb{C}$ in the set $\{z = x + iy \mid x \geq c\}$ where $c > 1$, we have

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \leq \sum_{n=1}^{\infty} \frac{M}{n^z} = \sum_{n=1}^{\infty} \frac{M}{n^x} \leq \sum_{n=1}^{\infty} \frac{M}{n^c} < \infty$$

so that $\sum_{n=1}^{\infty} \frac{a_n}{n^z}$ converges.

10 Show that $f_n(x) = n^3 x^n (1-x)$ does not converge uniformly on $[0,1]$.

For $x_n = 1 - 1/n$ (which is always in $[0,1]$) then

$$f_n(x_n) = n^3 (1-1/n)^n (1-(1-1/n)) = n^2 (1-1/n)^n$$

Therefore since $(1-1/n)^n \to 1/e$ as $n \to \infty$ then f_n can be made arbitrarily large and so cannot converge uniformly.

11 Give an example of a sequence of continuous functions for each of the below.

(a) A sequence of continous functions that converges to zero on $[0, 1]$ but not uniformly

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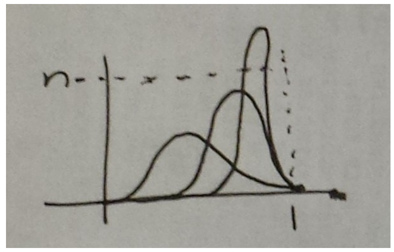
Problem ten has already proven this.

(b)



(c)