Math 508: Advanced Analysis
Homework 11

Lawrence Tyler Rush
<me@tylerlogic.com>

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1

Let $L : X \to Y$ be linear map between vector spaces X and Y such that $x_1, x_2 \in X$ are solutions, respectively, to

$$Lx = y_1 \qquad \text{and} \qquad Lx = y_2$$

for some $y_1, y_2 \in Y$. Furthermore, let $z \neq 0$ be a solution to $Lx = 0$.

(a) Find a solution for $Lx = 3y_1$

The vector $3x_1$ is a solution since $L(3x_1) = 3(Lx_1) = 3y_1$.

(b) Find a solution for $Lx = -5y_2$

The vector $-5x_2$ is a solution since $L(-5x_2) = -5(Lx_2) = -5y_2$.

(c) Find a solution for $Lx = 3y_1 - 5y_2$

The vector $3x_1 - 5x_2$ is a solution since $L(3x_1 - 5x_2) = 3(Lx_1) - 5(Lx_2) = 3y_1 - 5y_2$.

(d) Find a solution other than z and 0 for $Lx = 0$

The vector $2z$ is a solution since $L(2z) = 2(Lz) = 2(0) = 0$.

(e) Find two solutions of $Lx = y_1$

Both $x_1 + z$ and $x_1 + 2z$ are solutions since $L(x_1 + z) = Lx_1 + Lz = y_1 + 0 = y_1$ and $L(x_1 + 2z) = Lx_1 + 2Lz = y_1 + 2(0) = y_1$.

(f) Find another solution to $Lx = 3y_1 - 5y_2$

The vector $3x_1 - 5x_2 + z$ is another solution since $L(3x_1 - 5x_2 + z) = 3(Lx_1) - 5(Lx_2) + Lz = 3y_1 - 5y_2 + 0 = 3y_1 - 5y_2$.

2

3

(a)

(b)

4

5

Since $K(x, y)$ is continuous on $[0,1] \times [0,1]$ and $[0,1] \times [0,1]$ is compact in $\mathbb{R}^2$, then K is uniformly continuous on $[0,1]\times[0,1]$. Let $\varepsilon > 0$. Then for any $x, z \in \mathbb{R}$ there exists a $\delta > 0$ such that $|x-z| < \delta$ implies $|K(x,y)-K(z,y)| < \varepsilon$ for all $y \in \mathbb{R}$. This implies that

$$|h(x) - h(z)| = \left| \int_0^1 (K(x,y) - K(z,y))\, dy \right| \leq \int_0^1 |K(x,y) - K(z,y)|\, dy < \int_0^1 \varepsilon dy = \varepsilon$$

so that h is continuous.

6

If we rewrite

$$\int_0^1 e^{x-y} u(y) dy$$

as αe^x where we define the constant $\alpha = \int_0^1 e^{-y} u(y) dy$ we can simplify $u(x)$ as

$$u(x) = f(x) + \lambda \alpha e^x \tag{6.1}$$

From this simplification we have $e^{-x}u(x) = e^{-x}f(x) + \lambda\alpha$ which, after integrating both sides with respect to x over $[0,1]$, implies

$$\begin{aligned} \int_0^1 e^{-x} u(x) dx &= \int_0^1 e^{-x} f(x) dx + \int_0^1 \lambda \alpha dx \\ \alpha &= \int_0^1 e^{-x} f(x) dx + \lambda \alpha \\ \alpha &= \frac{1}{1-\lambda} \int_0^1 e^{-x} f(x) dx \end{aligned}$$

Combining the last line of the above equation with equation 6.1 we end up with

$$u(x) = f(x) + e^x \frac{\lambda}{1-\lambda} \int_0^1 e^{-y} f(y) dy$$

Hence we have a unique solution for

$$u(x) = f(x) + \lambda \int_0^1 e^{x-y} u(y) dy$$

so long as $\lambda \neq 0$.

7

Not worded properly.

8