Math 509: Advanced Analysis
Homework 3

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http://coursework.tylerlogic.com/courses/upenn/math509/homework03

The following solutions correspond to Frank Jones' book, chapter three: `http://www.owlnet.rice.edu/~fjones`

1 Problem 3-18

2 Problem 3-20

Let $f : R^2 \to R$ be defined as

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 + 2xy + 5x + y$$

This means the partial derivatives are

$$f_x = x^2 + 2y + 5 \quad \text{and} \quad f_y = y + 2x + 1$$

so that f_x is zero at points satisfying

$$y = \frac{-1}{2}(x^2 + 5)$$

and f_y is zero at all points satisfying

$$y = -2x - 1$$

These constraints yield two critical points, $(1,-3)$ and $(3,-7)$. The Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2x & 2 \\ 2 & 1 \end{pmatrix}$$

Therefore

$$\det(H(1,-3)) = \det \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = -2$$

indicating that $(1,-3)$ is a saddle point. Furthermore

$$\det(H(3,-7)) = \det \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix} = 2$$

indicating that $(3,-7)$ is a local minimum since the diagonals of the above Hessian matrix at $(3,-7)$ are possitive.

3 Problem 3-21

Let $f : R^2 \to R$ be defined as

$$f(x,y) = x^2y^2 - 5x^2 - 8xy - 5y^2$$

This means the partial derivatives are

$$f_x = 2xy^2 - 10x - 8y \quad \text{and} \quad f_y = 2x^2y - 8x - 10y$$

so that f_x is zero at points satisfying

$$x = 4y(y^2 - 5)^{-1} \tag{3.1}$$

Substituting this into the equation $0 = 2x^2y - 8x - 10y$ which is obtained by setting $f_y = 0$, we have the following sequence of equations

$$\begin{aligned}
2y(4y(y^2-5)^{-1})^2 - 8(4y(y^2-5)^{-1}) - 10y &= 0 \\
32y^3(y^2-5)^{-2} - 32y(y^2-5)^{-1} - 10y &= 0 \\
16y^3(y^2-5)^{-2} - 16y(y^2-5)^{-1} - 5y &= 0 \\
16y^3 - 16y(y^2-5) - 5y(y^2-5)^2 &= 0 \\
y\left(16y^2 - 16(y^2-5) - 5(y^2-5)^2\right) &= 0 \\
y\left(16y^2 - 16y^2 + 5(16) - 5(y^2-5)^2\right) &= 0 \\
y\left(5(16) - 5(y^2-5)^2\right) &= 0 \\
y\left(16 - (y^2-5)^2\right) &= 0 \\
y\left(4 + (y^2-5)\right)\left(4 - (y^2-5)\right) &= 0 \\
y\left(y^2-1\right)\left(9-y^2\right) &= 0
\end{aligned}$$

Thus f_y is zero whenever $y = 0, \pm 1, \pm 3$. By plugging these values back into equation 3.1 yields critical points at $(0,0)$, $(1,-1)$, $(-1,1)$, $(3,3)$, and $(-3,-3)$. The Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2y^2 - 10 & 4xy - 8 \\ 4xy - 8 & 2x^2 - 10 \end{pmatrix}$$

With this, we have the following values of the Hessian matrix at the critical points:

$$\det(H(0,0)) = \det \begin{pmatrix} -10 & -8 \\ -8 & -10 \end{pmatrix} = 36$$

$$\det(H(1,-1)) = \det \begin{pmatrix} -8 & -12 \\ -12 & -8 \end{pmatrix} = -80$$

$$\det(H(-1,1)) = \det \begin{pmatrix} -8 & -12 \\ -12 & -8 \end{pmatrix} = -80$$

$$\det(H(3,3)) = \det \begin{pmatrix} 8 & 28 \\ 28 & 8 \end{pmatrix} = -720$$

$$\det(H(-3,-3)) = \det \begin{pmatrix} 8 & 28 \\ 28 & 8 \end{pmatrix} = -720$$

indicating that $(0,0)$ is a local maximum whilst the other four critical points are saddle points.

4 Problem 3-22

Let $f : R^2 \to R$ be defined as

$$f(x,y) = xy(12 - 3x - 4y)$$

This means the partial derivatives are

$$f_x = y(12 - 6x - 4y) \quad \text{and} \quad f_y = x(12 - 3x - 8y)$$

so that f_x is zero whenever $y = 0$ or when (x,y) satisfies

$$3x = 6 - 2y$$

and f_y is zero whenever $x = 0$ or when (x,y) satisfies

$$3x = 12 - 8y$$

Combining these four constraints yields critical points of $(0,0)$, $(0,3)$, $(4,0)$, and $(4/3,1)$. The Hessian matrix is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -6y & 12-6x-8y \\ 12-6x-8y & -8x \end{pmatrix}$$

With this, we have the following values of the Hessian matrix at the critical points:

$$\det(H(0,0)) = \det \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} = -144$$

$$\det(H(0,3)) = \det \begin{pmatrix} -12 & -12 \\ -12 & 0 \end{pmatrix} = 144$$

$$\det(H(4,0)) = \det \begin{pmatrix} 0 & -12 \\ -12 & -32 \end{pmatrix} = 144$$

$$\det(H(4/3,1)) = \det \begin{pmatrix} -6 & -4 \\ -4 & -32/3 \end{pmatrix} = 48$$

indicating that $(0,0)$ is a saddle point and the remaining three points are local minimums.

5 Problem 3-23

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6 Problem 3-24

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7 Problem 3-27

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8 Problem 3-28
