Math 509: Advanced Analysis
Homework 4

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1 Problem 8 from slides

Define $f : R^2 \to R^2$ to be $f(x,y) = (e^x + e^y, e^x + e^{-y})$. Then

$$f_1(x,y) = e^x + e^y \qquad \text{and} \qquad f_2(x,y) = e^x + e^{-y}$$

The Jacobian matrix of f at a point (x,y), call it J, is

$$J = \begin{pmatrix} \frac{d}{dx}f_1 & \frac{d}{dy}f_1 \\ \frac{d}{dx}f_2 & \frac{d}{dy}f_2 \end{pmatrix} = \begin{pmatrix} e^x & e^y \\ e^x & -e^{-y} \end{pmatrix}$$

Then $\det(J) = -e^x(e^{-y} - e^y)$, implying that J is not invertible whenever $e^{-y} - e^y = 0$, i.e. whenever $y = 0$. Since f is continuously differentiable, then the inverse function theorem tells us that f is locally invertible whenever $y \neq 0$ since J will be invertible at those points.

Compute the Jacobian matrix of the inverse map

The Jacobian matrix of the inverse map is inverse of the Jacobian matrix of f. Now since we computed the Jacobian matrix of f, call it J, at arbitrary (x,y), then given

$$\begin{pmatrix} -e^{-x}e^y(e^{-y}+e^y)^{-1} + e^{-x} & e^{-x}e^y(e^{-y}+e^y)^{-1} \\ (e^{-y}+e^y)^{-1} & -(e^{-y}+e^y)^{-1} \end{pmatrix} \begin{pmatrix} e^x & e^y \\ e^x & -e^{-y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} e^x & e^y \\ e^x & -e^{-y} \end{pmatrix} \begin{pmatrix} -e^{-x}e^y(e^{-y}+e^y)^{-1} + e^{-x} & e^{-x}e^y(e^{-y}+e^y)^{-1} \\ (e^{-y}+e^y)^{-1} & -(e^{-y}+e^y)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$J^{-1} = \begin{pmatrix} -e^{-x}e^y(e^{-y}+e^y)^{-1} + e^{-x} & e^{-x}e^y(e^{-y}+e^y)^{-1} \\ (e^{-y}+e^y)^{-1} & -(e^{-y}+e^y)^{-1} \end{pmatrix}$$

is the Jacobian of the inverse of f at the point $f(x,y)$.

2 Problem 9 from slides

Define $f : R^2 \to R^2$ to be $f(x,y) = (e^x + e^y, e^x - e^y)$. Define $g : S \to R^2$ where $S = f(R^2) = \{(u,v) \in R^2 \mid u > v,\ u > 0\}$ by

$$g(u,v) = \left(\ln\left(\frac{1}{2}(u+v)\right), \ln\left(\frac{1}{2}(u-v)\right)\right)$$

Now because

$$\begin{aligned} g(f(x,y)) &= g(e^x + e^y, e^x - e^y) \\ &= \left(\ln\left(\frac{1}{2}(e^x + e^y + e^x - e^y)\right), \ln\left(\frac{1}{2}(e^x + e^y - (e^x - e^y))\right)\right) \\ &= \left(\ln\left(\frac{1}{2}(2e^x)\right), \ln\left(\frac{1}{2}(2e^y)\right)\right) \\ &= (\ln(e^x), \ln(e^y)) \\ &= (x,y) \end{aligned}$$

and

$$\begin{aligned} f(g(u,v)) &= f\left(\ln\left(\frac{1}{2}(u+v)\right), \ln\left(\frac{1}{2}(u-v)\right)\right) \\ &= \left(e^{\ln\left(\frac{1}{2}(u+v)\right)} + e^{\ln\left(\frac{1}{2}(u-v)\right)}, e^{\ln\left(\frac{1}{2}(u+v)\right)} - e^{\ln\left(\frac{1}{2}(u-v)\right)}\right) \\ &= \left(\left(\frac{1}{2}(u+v)\right) + \left(\frac{1}{2}(u-v)\right), \left(\frac{1}{2}(u+v)\right) - \left(\frac{1}{2}(u-v)\right)\right) \\ &= (u,v) \end{aligned}$$

then $g = f^{-1}$. Because

$$g_1(u,v) = \ln\left(\frac{1}{2}(u+v)\right) \qquad \text{and} \qquad g_2(u,v) = \ln\left(\frac{1}{2}(u-v)\right)$$

then the Jacobian of g, denote it by J_g, is

$$J_g = \begin{pmatrix} \frac{d}{du}g_1 & \frac{d}{dv}g_1 \\ \frac{d}{du}g_2 & \frac{d}{dv}g_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{u+v} & \frac{1}{u+v} \\ \frac{1}{u-v} & \frac{-1}{u-v} \end{pmatrix}$$

Furthermore, because

$$f_1(x,y) = e^x + e^y \qquad \text{and} \qquad f_2(x,y) = e^x - e^y$$

then the Jacobian of f, denote it by J_f, is

$$J_f = \begin{pmatrix} \frac{d}{dx}f_1 & \frac{d}{dy}f_1 \\ \frac{d}{dx}f_2 & \frac{d}{dy}f_2 \end{pmatrix} = \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix}$$

Finally, to check that indeed J_f and J_g are inverses of each other, we perform the following two multiplications for these matrices evaluated at the points $(u,v) = f(x,y)$ and (x,y), respectively and then ensure the result of each is the identity matrix.

$$\begin{aligned} J_gJ_f &= \begin{pmatrix} \frac{1}{u+v} & \frac{1}{u+v} \\ \frac{1}{u-v} & \frac{-1}{u-v} \end{pmatrix} \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{(e^x+e^y)+(e^x-e^y)} & \frac{1}{(e^x+e^y)+(e^x-e^y)} \\ \frac{1}{(e^x+e^y)-(e^x-e^y)} & \frac{-1}{(e^x+e^y)-(e^x-e^y)} \end{pmatrix} \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2e^x} & \frac{1}{2e^x} \\ \frac{1}{2e^y} & \frac{-1}{2e^y} \end{pmatrix} \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} J_fJ_g &= \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \begin{pmatrix} \frac{1}{u+v} & \frac{1}{u+v} \\ \frac{1}{u-v} & \frac{-1}{u-v} \end{pmatrix} \\ &= \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \begin{pmatrix} \frac{1}{(e^x+e^y)+(e^x-e^y)} & \frac{1}{(e^x+e^y)+(e^x-e^y)} \\ \frac{1}{(e^x+e^y)-(e^x-e^y)} & \frac{-1}{(e^x+e^y)-(e^x-e^y)} \end{pmatrix} \\ &= \begin{pmatrix} e^x & e^y \\ e^x & -e^y \end{pmatrix} \begin{pmatrix} \frac{1}{2e^x} & \frac{1}{2e^x} \\ \frac{1}{2e^y} & \frac{-1}{2e^y} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

3 Problem 10 from slides

Define $f : R^3 \to R^3$ by $f(x_1, x_2, x_3) = (y_1, y_2, y_3)$ where

$$\begin{aligned} y_1 &= x_1 + x_2^2 + (x_3 - 1)^4 \\ y_2 &= x_1^2 + x_2 + (x_3^3 - 3x_3) \\ y_3 &= x_1^3 + x_2^3 + x_3 \end{aligned}$$

Then the Jacobian matrix of f at a point (x_1, x_2, x_3) is

$$\begin{pmatrix} 1 & 2x_2 & 4x_3 - 4 \\ 2x_1 & 1 & 3x_3^2 - 3 \\ 3x_1^2 & 3x_2 & 1 \end{pmatrix}$$

which is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

at the point $(0, 0, 1)$. The matrix above has determinant of 1 (it's the identity matrix). Hence $f'(0, 0, 1)$ is invertible. Furthermore, the fact that each of y_1, y_2, and y_3 are polynomials makes f continuously differentiable. These two facts therefore tell us that f is locally invertable at $(0, 0, 1)$, by the inverse function theorem.

4 Problem 11 from slides

Let $f : R \to R$ be a C^1 function with the additional property that

$$|f'(t)| \leq c < 1 \tag{4.1}$$

for all t and some real value c. Define $F : R^2 \to R^2$ by

$$F(x, y) = (x + f(y), y + f(x))$$

Prove F is injective

Assume that $(x, y), (x', y') \in R^2$ such that $F(x, y) = F(x', y')$. We then have

$$x + f(y) = x' + f(y') \qquad \text{and} \qquad y + f(x) = y' + f(x')$$

which can be rewritten as

$$x' - x = f(y) - f(y') \qquad \text{and} \qquad y' - y = f(x) - f(x')$$

The property in 4.1 then informs us that

$$|x - x'| = |f(y) - f(y')| \leq c|y - y'| \tag{4.2}$$

$$|y - y'| = |f(x) - f(x')| \leq c|x - x'| \tag{4.3}$$

Now substituting $|y - y'|$ from equation 4.3 into equation 4.2 yields

$$|x - x'| \leq c^2|x - x'|$$

Therefore $|x - x'| = |x - x'|$ (whether or not $c = 0$), which implies $x = x'$. Furthermore we can similarly conclude $y = y'$ using the same argument after subsituting $|x - x'|$ from equation 4.2 into equation 4.3. Hence we have $(x, y) = (x', y')$, and since our only assumption was that $F(x, y) = F(x', y')$, then F is injective.

Prove F is surjective

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5 Outline and describe the steps and ideas in Rudin's proof of the Inverse Function Theorem, without giving all the details, to the extent that satisfies you personally.

I'm not really satisfied, personally, with a proof that is missing details, so I will provide a full proof.

Let $f : R \to R^n$ be a continuously differentiable mapping of an open set E, and $f'(a)$ be invertible for some $a \in E$.

Define an open set $U \subset E$ and prove f is injective on it

Denote $f'(a)$ by just A, and choose λ such that

$$2\lambda|A^{-1}| = 1 \tag{5.4}$$

Note that this is not ill-defined as A is invertible by the hypothesis. Furthermore f' is continuous at a, being that f is continuously differentiable, which implies that there is an open ball $U \subset E$ centered at a such that

$$|f'(x) - A| < \lambda \tag{5.5}$$

for all $x \in U$.

We now create a helpful function that makes our proof easier; followed by a helpful lemma. For each $y \in R^n$ define $\varphi_y : R^n \to R^n$ by $\varphi_y(x) = x - A^{-1}(y - f(x))$.

Remark 5.1. Note that with this deifintion, we have that x is a fixed point of φ_y if and only if $f(x) = y$.

It also turns out that this function is a contraction mapping.

Lemma 5.1. *For any $y \in R^n$, the function φ_y has that*

$$|\varphi_y(x_1) - \varphi(x_2)| < \frac{1}{2}|x_1 - x_2|$$

for all $x_1, x_2 \in U$.

Proof. We see that the derivative of φ_y is $\varphi_y'(x) = I - A^{-1}f'(x) = A^{-1}A - A^{-1}f'(x) = A^{-1}(A - f'(x))$. Equations 5.4 and 5.5 thus imply $|\varphi_y'(x)| = |A^{-1}||A - f'(x)| < |A^{-1}|\lambda = \frac{1}{2}$. Therefore Rudin's theorem 9.19 tells us that

$$|\varphi_y(x_1) - \varphi(x_2)| < \frac{1}{2}|x_1 - x_2|$$

for all $x_1, x_2 \in U$, as desired. □

Becuase lemma 5.1 indicates that φ_y is a contraction mapping, then it can have at most one fixed point. In light of remark 5.1, we then know that f must be injective on U. We next move on to proving that $f(U)$ is open.

Prove $V = f(U)$ is open

Denote $f(U)$ by V and choose a $y_0 \in V$. Since f is certainly onto its image, then we of course have the existence of some $x_0 \in U$ with $f(x_0) = y_0$. Define B to be an open ball centered at x_0 with a radius $r > 0$ small enough so that the closure of B, denote it by $\overline{B}$, is contained in U. To show the openess of V, we show that the open ball of radius λr centered at y_0 is contained in V. So let $y \in V$ such that $|y - y_0| < \lambda r$. Then we have

$$|\varphi_y(x_0) - x_0| = |A^{-1}(y - y_0)| = |A^{-1}||y - y_0| < |A^{-1}|\lambda r = \frac{1}{2}r$$

by the definition of λ in equation 5.4. Thus if $x \in \overline{B}$, Lemma 5.1 paired with the above equation imply

$$|\varphi_y(x) - x_0| = |\varphi_y(x) - \varphi(x_0) + \varphi(x_0) - x_0| \leq |\varphi_y(x) - \varphi(x_0)| + |\varphi(x_0) - x_0| < \frac{1}{2}|x - x_0| + \frac{1}{2}r < \frac{1}{2}r + \frac{1}{2}r = r$$

Hence $\varphi_y(x) \in B$, i.e. φ_y is a contraction of $\overline{B}$ into itself. Since $\overline{B}$ is closed, it is complete, and the contraction mapping principle (Rudin Theorem 9.23) therefore implies the existence of one and only one fixed point of φ_y, which Remark 5.1 tells us is an $x \in \overline{B}$ where $f(x) = y$. In other words, $y \in f(\overline{B}) \subset f(U) = V$. Hence V is open.