Math 509: Advanced Analysis
Homework 5

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1 Problem 12 from Slides

Define $f : R^4 \to R^3$ by

$$f(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u)$$

With this we can define four functions on $R \times R^3$:

$$f_1(x, (y, z, u)) = f(x, y, z, u) \tag{1.1}$$
$$f_2(y, (x, z, u)) = f(x, y, z, u) \tag{1.2}$$
$$f_3(z, (x, y, u)) = f(x, y, z, u) \tag{1.3}$$
$$f_4(u, (x, y, z)) = f(x, y, z, u) \tag{1.4}$$

Thus we have $f_i(0, (0, 0, 0)) = f(0, 0, 0, 0) = (0, 0, 0, 0)$ for all $i = 1, 2, 3, 4$. Denote the second argument of each of f_i by v, then we have the following derivatives

$$(\partial f_1/\partial v)_{(0,(0,0,0))} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix}$$
$$(\partial f_2/\partial v)_{(0,(0,0,0))} = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{pmatrix}$$
$$(\partial f_3/\partial v)_{(0,(0,0,0))} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$
$$(\partial f_4/\partial v)_{(0,(0,0,0))} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{pmatrix}$$

with the corresponding determinants

$$\det\left((\partial f_1/\partial v)_{(0,(0,0,0))}\right) = 1(4 - (-3)) - (-1)(-2 - 2) + (0)(3 - 4) = 11$$
$$\det\left((\partial f_2/\partial v)_{(0,(0,0,0))}\right) = 3(4 - (-3)) - (-1)(2 - 2) + (0)(-3 - 4) = 21$$
$$\det\left((\partial f_3/\partial v)_{(0,(0,0,0))}\right) = 3(-1 - 2) - (1)(2 - 2) + (0)(2 - (-2)) = -9$$
$$\det\left((\partial f_4/\partial v)_{(0,(0,0,0))}\right) = 3(3 - 4) - (1)(-3 - 4) + (-1)(2 - (-2)) = 0$$

Therefore because $(\partial f_1/\partial v)_{(0,(0,0,0))}$, $(\partial f_2/\partial v)_{(0,(0,0,0))}$, and $(\partial f_3/\partial v)_{(0,(0,0,0))}$ all have nonzero determinant, then the implicit function theorem tells us there exist functions $g_1 : R \to R^3$, $g_2 : R \to R^3$, and $g_3 : R \to R^3$ such that

$$f_1(0, g_1(0)) = f_1(0, (0, 0, 0)) = (0, 0, 0)$$
$$f_2(0, g_2(0)) = f_2(0, (0, 0, 0)) = (0, 0, 0)$$
$$f_3(0, g_3(0)) = f_3(0, (0, 0, 0)) = (0, 0, 0)$$

In light of the defintion of f and equations 1.1 through 1.3, this implies that the equations

$$3x + y - z + u^2 = 0 \tag{1.5}$$
$$x - y + 2z + u = 0 \tag{1.6}$$
$$2x + 2y - 3z + 2u = 0 \tag{1.7}$$

have solutions for

1. y, z, u in terms of x, due to g_1
2. x, z, u in terms of y, due to g_2
3. x, y, u in terms of z, due to g_3

On the other hand, because the determinant of $(\partial f_4/\partial v)_{(0,(0,0,0))}$ is zero then there is no function $g_4 : R \to R^3$ with $f_4(0, g_4(0))$, implying that there is no solution to equations 1.5 through 1.7 for x, y, z in terms of u.

2 Problem 13 from Slides

Define $f : R^2 \times R \to R$ by

$$f(x, y, z) = z^2 x + e^z + y$$

Then because

$$\det\left(\frac{\partial f}{\partial z}\right)_{(1,-1,0)} = \det\left(2xz + e^z\right)_{(1,-1,0)} = \det(1) = 1$$

the implicit function theorem tells us that there exists a $g : R^2 \to R$ such that $g(1, -1) = 0$ and $f(x, y, g(x, y)) = 0$ for all (x, y) in some neighborhood of $(1, -1)$. Furthermore, since

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial g}\right)\left(\frac{\partial g}{\partial x}\right)$$

and we know that

$$\frac{\partial f}{\partial x} = z^2$$

and

$$\left(\frac{\partial f}{\partial g}\right) = 2gx + e^g$$

then we can solve for $\frac{\partial g}{\partial x}$ to get

$$\frac{\partial g}{\partial x} = z^2(2gx + e^g)^{-1}$$

thus

$$\frac{\partial g}{\partial x}(1, -1) = z^2(2g(1, -1) + e^{g(1,-1)})^{-1} = z^2$$

Similarly we have

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial g}\right)^{-1} = \left(2g(x, y)x + e^{g(x,y)}\right)$$

so that

$$\frac{\partial g}{\partial y}(1, -1) = 1$$

3 Problem 14 from Slides

Define a function $f : R \times R^3 \to R^3$ by

$$f(t, (x, y, z)) = (t^2 + x^3 + y^3 + z^3, t + x^2 + y^2 + z^2, t + x + y + z)$$

Let u denote the second component in R^3 of $f(t, u)$. Then the partial derivative with respect to u at the point $(t, u) = (0, (-1, 1, 0))$ is

$$\left[\frac{\partial f}{\partial u}\right]_{(0,(-1,1,0))} = \begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}_{(0,(-1,1,0))} = \begin{pmatrix} 3 & 3 & 0 \\ -2 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

which has determinant of $3(2 - 0) - 3(-2 - 0) + 0 = 12$. The above matrix is therefore nonsingular, and hence the Implicit Function Theorem yields the existence of a function $g : R \to R^3$ defined on a neighborhood of 0 such that $g(0) = (-1, 1, 0)$ and $f(t, g(t)) = f(0, (-1, 1, 0)) = (0, 2, 0)$ for all points in that neighborhood. In turn, g implicitly defines three real-valued functions on R, x, y, z, where $g(t) = (x(t), y(t), z(t))$. With this notation, the previously mentioned result of the the Implicit Function Theorem can be restated as: there exists a neighborhood around $(0, -1, 1, 0)$ such that $f(t, (x(t), y(t), z(t))) = (0, 2, 0)$ for all t. In other words, given the definition of f, the following equations have solutions around $(t, x, y, z) = (0, -1, 1, 0)$

$$\begin{aligned} t^2 + (x(t))^3 + (y(t))^3 + (z(t))^3 &= 0 \\ t + (x(t))^2 + (y(t))^2 + (z(t))^2 &= 2 \\ t + x(t) + y(t) + z(t) &= 0 \end{aligned}$$

4 Problem 15 from Slides

Define three functions $f', g', h' : R^2 \times R \to R$ by

$$\begin{aligned} f'((x,y),z) &= xz + \sin(xy) + \cos(xz) \\ g'((y,z),x) &= xz + \sin(xy) + \cos(xz) \\ h'((x,z),y) &= xz + \sin(xy) + \cos(xz) \end{aligned}$$

then we have the following derivatives

$$\begin{aligned} f'_z &= z + y\cos(xy) - z\sin(xz) \\ g'_x &= x\cos(xy) \\ h'_y &= x + x\cos(xz) \end{aligned}$$

so that

$$\begin{aligned} f'_z(0,1,1) &= 2 \\ g'_x(0,1,1) &= 0 \\ h'_y(0,1,1) &= 0 \end{aligned}$$

Thus the Implicit Function Theorem tells us that there is a differentiable function $f : R^2 \to R$ such that $z = f(x,y)$ but that there are no such functions $g, h : R^2 \to R$ with $x = g(y,z)$ or $y = h(x,z)$

5 Problem 16 from Slides

Let $F(x,y)$ be a C^2 function such that $F(x, f(x)) = 0$ and $(\partial F/\partial y)(x, f(x)) \neq 0$ for all $x \in R$. Then we have that

$$\frac{\partial F}{\partial x}(x, f(x)) = \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y} f' \tag{5.8}$$

By assumption $\frac{\partial F}{\partial y} \neq 0$ so that we may solve for f' as

$$f' = \left(\frac{\partial F}{\partial y}\right)^{-1} \left(\frac{\partial F}{\partial x}(x, f(x)) - \frac{\partial F}{\partial x}(x,y)\right) \tag{5.9}$$

Again taking the derivative with respect to x in equation 5.8 yields

$$\frac{\partial^2 F}{\partial^2 x}(x, f(x)) = \frac{\partial^2 F}{\partial^2 x}(x,y) + \frac{\partial^2 F}{\partial^2 y} f' + \frac{\partial F}{\partial y} f''$$

substituting equation 5.9 into this equation and then solving for f'' leaves us with

$$f'' = \left(\frac{\partial F}{\partial y}\right)^{-1} \left(\frac{\partial^2 F}{\partial^2 x}(x, f(x)) - \frac{\partial^2 F}{\partial^2 x}(x,y) - \frac{\partial^2 F}{\partial^2 y}\left(\frac{\partial F}{\partial y}\right)^{-1}\left(\frac{\partial F}{\partial x}(x, f(x)) - \frac{\partial F}{\partial x}(x,y)\right)\right)$$

6 Problem 17 from Slides

Define $F : R^3 \to R$ by

$$F(x,y,z) = x^2 + 4y^2 - 2yz - z^2$$

(a) Verify the hypotheses of the Implicit Function Theorem

(b) Find the largest neighborhood U of $(2, 1, -4)$ on which $\partial F/\partial z \neq 0$

We have that

$$\frac{\partial F}{\partial z} = -2y - 2z = -2(y + z)$$

which means that $\frac{\partial F}{\partial z} \neq 0$ whenever $y \neq -z$. Hence any point of the form $(x, y, -y)$ will have $\frac{\partial F}{\partial z} = 0$ which implies that the largest neighborhood of $(2, 1, -4)$ will be the open ball with radius r where r is the distance from $(2, 1, -4)$ to the closest point of the form $(x, y, -y)$. Since $y \neq -z$ at the point $(2, 1, -4)$, we know that such an r will exist. We just need to find it.

To do so, we can find the minimum of the distance between $(2, 1, -4)$ and any point $(x, y, -y)$, or equivalently the minimum of the square of the distance between $(2, 1, -4)$ and any point $(x, y, -y)$. So define $f : R^2 \to R$ by

$$f(x, y) = (x - 2)^2 + (y - 1)^2 + (-y - (-4))^2 = x^2 - 4x + 4 + y^2 - 2y + 1 + y^2 + 8y + 16 = x^2 - 4x + 2y^2 + 6y + 21$$

With this definition we have

$$\begin{aligned} f_x &= 2x - 4 \\ f_y &= 2y + 6 \end{aligned}$$

indicating that there is a critical point at $(x, y) = (2, -3)$, which we simply need to confirm that this is indeed a minimum. We compute the Hessian matrix of this function

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

for which $\det H = 4$ so that $(2, -3)$ is indeed a minimum since the diagonals of H are both positive and its determinant is positive. Therefore

$$r^2 = 2^2 - 4(2) + 2(-3)^2 + 6(-3) + 21 = 17$$

so that $r = \sqrt{17}$. Hence we set U to be the open ball of radius $\sqrt{17}$ centered at $(2, 1, -4)$.

(c)