Math 509: Advanced Analysis
Homework 8

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For the first five problems, let $A, B, C, D \in R^3$ with $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$, and $D = (d_1, d_2, d_3)$.

1 Problem 1 from Vector Calculus Notes

First off, we have

$$\begin{aligned}
B \times C &= (b_2c_3 - b_3c_2,\ b_1c_3 - b_3c_1,\ b_1c_2 - b_2c_1) \\
C \times A &= (c_2a_3 - c_3a_2,\ c_1a_3 - c_3a_1,\ c_1a_2 - c_2a_1) \\
A \times B &= (a_2b_3 - a_3b_2,\ a_1b_3 - a_3b_1,\ a_1b_2 - a_2b_1)
\end{aligned}$$

so that

$$\begin{aligned}
A \cdot (B \times C) &= a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\
B \cdot (C \times A) &= b_1(c_2a_3 - c_3a_2) + b_2(c_1a_3 - c_3a_1) + b_3(c_1a_2 - c_2a_1) \\
&= a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\
C \cdot (A \times B) &= c_1(a_2b_3 - a_3b_2) + c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1) \\
&= a_1(b_2c_3 - b_3c_2) + a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)
\end{aligned}$$

2 Problem 2 from Vector Calculus Notes

We have

$$\begin{aligned}
A \times (B \times C) = \Big(& a_2(b_1c_2 - b_2c_1) - a_3(b_1c_3 - b_3c_1), \\
& a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2), \\
& a_1(b_1c_3 - b_3c_1) - a_2(b_2c_3 - b_3c_2) \Big)
\end{aligned}$$

We also have

$$A \cdot C = a_1c_1 + a_2c_2 + a_3c_3$$

and

$$A \cdot B = a_1b_1 + a_2b_2 + a_3b_3$$

so that

$$
\begin{aligned}
B(A\cdot C)-C(A\cdot B) = \Big(&(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1,\\
&(a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2,\\
&(a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\Big)\\
=\Big(&a_1b_1c_1+a_2b_1c_2+a_3b_1c_3-a_1b_1c_1-a_2b_2c_1-a_3b_3c_1,\\
&a_1b_2c_1+a_2b_2c_2+a_3b_2c_3-a_1b_1c_2-a_2b_2c_2-a_3b_3c_2,\\
&a_1b_3c_1+a_2b_3c_2+a_3b_3c_3-a_1b_1c_3-a_2b_2c_3-a_3b_3c_3\Big)\\
=\Big(&a_2b_1c_2+a_3b_1c_3-a_2b_2c_1-a_3b_3c_1,\\
&a_1b_2c_1+a_3b_2c_3-a_1b_1c_2-a_3b_3c_2,\\
&a_1b_3c_1+a_2b_3c_2-a_1b_1c_3-a_2b_2c_3\Big)\\
=\Big(&a_2(b_1c_2-b_2c_1)-a_3(b_1c_3-b_3c_1),\\
&a_1(b_1c_2-b_2c_1)-a_3(b_2c_3-b_3c_2),\\
&a_1(b_1c_3-b_3c_1)-a_2(b_2c_3-b_3c_2)\Big)
\end{aligned}
$$

Hence $A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$.

3 Problem 3 from Vector Calculus Notes

Let $A,B,C\in R^n$. Then in light of the previous problem and the fact that the dot product is a commutative operation, we have

$$
\begin{aligned}
A\times(B\times C)+B\times(C\times A)+C\times(A\times B) &= \big(B(A\cdot C)-C(A\cdot B)\big)+\big(C(B\cdot A)-A(B\cdot C)\big)+\big(A(C\cdot B)-B(C\cdot A)\big)\\
&= \big(B(A\cdot C)-B(C\cdot A)\big)+\big(C(B\cdot A)-C(A\cdot B)\big)+\big(A(C\cdot B)-A(B\cdot C)\big)\\
&=0
\end{aligned}
$$

4 Problem 4 from Vector Calculus Notes

Since

$$
\begin{aligned}
A\times(B\times C) = \Big(&a_2(b_1c_2-b_2c_1)-a_3(b_1c_3-b_3c_1),\\
&a_1(b_1c_2-b_2c_1)-a_3(b_2c_3-b_3c_2),\\
&a_1(b_1c_3-b_3c_1)-a_2(b_2c_3-b_3c_2)\Big)\\
=\Big(&a_2b_1c_2-a_2b_2c_1-a_3b_1c_3+a_3b_3c_1,\\
&a_1b_1c_2-a_1b_2c_1-a_3b_2c_3+a_3b_3c_2,\\
&a_1b_1c_3-a_1b_3c_1-a_2b_2c_3+a_2b_3c_2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
(A \times B) \times C &= (a_2b_3 - a_3b_2,\ a_1b_3 - a_3b_1,\ a_1b_2 - a_2b_1) \times C \\
&= \Big(c_2(a_1b_2 - a_2b_1) - c_3(a_1b_3 - a_3b_1), \\
&\quad c_1(a_1b_2 - a_2b_1) - c_3(a_2b_3 - a_3b_2), \\
&\quad c_1(a_1b_3 - a_3b_1) - c_2(a_2b_3 - a_3b_2) \Big) \\
&= \Big(a_1b_2c_2 - a_2b_1c_2 - a_1b_3c_3 + a_3b_1c_3, \\
&\quad a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3, \\
&\quad a_1b_3c_1 - a_3b_1c_1 - a_2b_3c_2 + a_3b_2c_2 \Big)
\end{aligned}
$$

Hence we have that $A \times (B \times C) = (A \times B) \times C$ whenever all of

$$
\begin{aligned}
a_2b_1c_2 - a_2b_2c_1 - a_3b_1c_3 + a_3b_3c_1 &= a_1b_2c_2 - a_2b_1c_2 - a_1b_3c_3 + a_3b_1c_3 \\
a_1b_1c_2 - a_1b_2c_1 - a_3b_2c_3 + a_3b_3c_2 &= a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3, \\
a_1b_1c_3 - a_1b_3c_1 - a_2b_2c_3 + a_2b_3c_2 &= a_1b_3c_1 - a_3b_1c_1 - a_2b_3c_2 + a_3b_2c_2
\end{aligned}
$$

are satisfied.

5 Problem 5 from Vector Calculus Notes

6 Problem 9 from Vector Calculus Notes

Let $\mathbf{r} = (x, y, z)$ be an element in R^3 and put $r = |\mathbf{r}|$ and $\hat{r} = \mathbf{r}/r$.

(a) Show $\nabla(r^2) = 2\mathbf{r}$

Since

$$r^2 = |\mathbf{r}|^2 = \left(\sqrt{x^2 + y^2 + z^2}\right)^2 = x^2 + y^2 + z^2$$

so that

$$\nabla(r^2) = (2x, 2y, 2z) = 2(x, y, z) = 2\mathbf{r}$$

(b) Show $\nabla(1/r) = -\hat{r}/r^2$

Since r is a function of x, y, and z, then

$$\begin{aligned}
\nabla\left(\frac{1}{r}\right) &= \nabla(r^{-1}) \\
&= \left(-r^{-2}\frac{dr}{dx}, -r^{-2}\frac{dr}{dy}, -r^{-2}\frac{dr}{dz}\right) \\
&= -r^{-2}\left(\frac{dr}{dx}, \frac{dr}{dy}, \frac{dr}{dz}\right) \\
&= -r^{-2}\left(\frac{1}{2r}(2x), \frac{1}{2r}(2y), \frac{1}{2r}(2z)\right) \\
&= -r^{-2}\left(\frac{1}{2r}\right)(2x, 2y, 2z) \\
&= -r^{-2}\left(\frac{1}{r}\right)(x, y, z) \\
&= -r^{-2}\left(\frac{1}{r}\right)\mathbf{r} \\
&= -r^{-2}\left(\frac{\mathbf{r}}{r}\right) \\
&= -\frac{\hat{r}}{r^2}
\end{aligned}$$

7 Problem 10 from Vector Calculus Notes

8 Problem 11 from Vector Calculus Notes

9 Problem 12 from Vector Calculus Notes

Define a vector field V in R^3 by

$$V(x, y, z) = (-y, x, 1)$$

Then for any φ_t in the group of one-parameter diffeomorphisms generated by this vector field must have

$$\frac{d}{dt}\varphi_t(x, y, z) = V(\varphi_t(x, y, z))$$

By defining $\varphi_t(x, y, z) = (x^*(t, x, y, z), y^*(t, x, y, z), z^*(t, x, y, z))$, the above equation yields

$$\begin{aligned}
\frac{d}{dt}(x^*(t, x, y, z), y^*(t, x, y, z), z^*(t, x, y, z)) &= V((x^*(t, x, y, z), y^*(t, x, y, z), z^*(t, x, y, z))) \\
&= (-y^*(t, x, y, z), x^*(t, x, y, z), 1)
\end{aligned}$$

further implying that

$$\begin{aligned}
\frac{d}{dt}x^* &= -y^* \\
\frac{d}{dt}y^* &= x^* \\
\frac{d}{dt}z^* &= 1
\end{aligned}$$

Hence the group $\{\varphi_t\}$ of one-parameter diffeomorphisms generated by V is the group of φ_t such that

$$\varphi_t(x,y,z) = \Big(f_t(x,y,z)\cos t,\ f_t(x,y,z)\sin t,\ t\Big)$$

where f_t is an arbitrary function on R^3 but fixed for each t.

Furthermore,

$$J\varphi_t = \begin{pmatrix} \cos t\frac{\partial f_t}{\partial x} & \cos t\frac{\partial f_t}{\partial y} & \cos t\frac{\partial f_t}{\partial z} \\ \sin t\frac{\partial f_t}{\partial x} & \sin t\frac{\partial f_t}{\partial y} & \sin t\frac{\partial f_t}{\partial z} \\ 0 & 0 & 0 \end{pmatrix}$$

which has determinant zero. Hence $\frac{d}{dt}|_{t=0} \det J\varphi_t$ is zero, and thus $\nabla \cdot V$ is also zero, indicating the $\{\varphi_t\}$ are volume preserving.

10 Problem 13 from Vector Calculus Notes

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11 Problem 14 from Vector Calculus Notes

Let

$$V = (x^2+y^2)^{-1}(-y\mathbf{i} + x\mathbf{j})$$

in other words

$$V(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

Therefore

$$\begin{aligned} \operatorname{curl} V &= \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}\frac{x}{x^2+y^2}, \frac{\partial}{\partial z}\frac{-y}{x^2+y^2} - \frac{\partial}{\partial x}0, \frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right) \\ &= \left(0, 0, \left(\frac{y^2+x^2}{(x^2+y^2)^2}\right) - \left(\frac{y^2+x^2}{(x^2+y^2)^2}\right)\right) \\ &= (0,0,0) \end{aligned}$$

as desired.

12 Problem 15 from Vector Calculus Notes
