Math 509: Advanced Analysis
Homework 9

Lawrence Tyler Rush
<me@tylerlogic.com>

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1 Problem 16 from Vector Calculus Notes

Let $A : R^3 \to R^3$, $B : R^3 \to R^3$, and $f : R^3 \to R$ and denote

$$A(x, y, z) = \Big(u(x, y, z), v(x, y, z), w(x, y, z)\Big)$$

and

$$B(x, y, z) = \Big(r(x, y, z), s(x, y, z), t(x, y, z)\Big)$$

(a) Derive Rule (1)

$$\begin{aligned}
\nabla(fg) &= (fg)_x + (fg)_y + (fg)_z \\
&= f_x g + f g_x + f_y g + f g_y + f_z g + f g_z \\
&= (f_x + f_y + f_z)g + f(g_x + g_y + g_z) \\
&= (\nabla f)g + f(\nabla g)
\end{aligned}$$

(b) Derive Rule (3)

$$\begin{aligned}
\nabla \cdot (fA) &= \nabla \cdot (fu, fv, fw) \\
&= (fu)_x + (fv)_y + (fw)_z \\
&= (f_x u + f u_x) + (f_y v + f v_y) + (f_z w + f w_z) \\
&= (f_x u + f_y v + f_z w) + (f u_x + f v_y + f w_z) \\
&= (f_x, f_y, f_z) \cdot A + f(u_x + v_y + w_z) \\
&= (\nabla f) \cdot A + f(\nabla \cdot A)
\end{aligned}$$

(c) Derive Rule (4)

$$\begin{aligned}
\nabla \cdot (A \times B) &= \nabla \cdot (vt - ws, wr - ut, us - vr) \\
&= (vt - ws)_x + (wr - ut)_y (us - vr)_z \\
&= v_x t + v t_x - w_x s - w s_x + w_y r + w r_y - u_y t - u t_y + u_z s + u s_z - v_z r - v r_z \\
&= \Big((w_y - v_z)r + (u_z - w_x)s + (v_x - u_y)t\Big) + \Big(u(s_z - t_y) + v(t_x - r_z) + w(r_y - s_x)\Big) \\
&= (w_y - v_z, u_z - w_x, v_x - u_y) \cdot B + A \cdot (s_z - t_y, t_x - r_z, r_y - s_x) \\
&= (w_y - v_z, u_z - w_x, v_x - u_y) \cdot B - A \cdot (t_y - s_z, r_z - t_x, s_x - r_y) \\
&= (\nabla \times A) \cdot B - A \cdot (\nabla \times B)
\end{aligned}$$

(d) Derive Rule (5)

$$
\begin{aligned}
\nabla \times (fA) &= \nabla \times (fu, fv, fw) \\
&= \Big((fw)_y - (fv)_z, (fu)_z - (fw)_x, (fv)_x - (fu)_y\Big) \\
&= \Big(f_y w + f w_y - f_z v - f v_z, f_z u + f u_z - f_x w - f w_x, f_x v + f v_x - f_y u - f u_y\Big) \\
&= \Big(f_y w - f_z v, f_z u - f_x w, f_x v - f_y u\Big) + \Big(f w_y - f v_z, f u_z - f w_x, f v_x - f u_y\Big) \\
&= (f_x, f_y, f_z) \times (u, v, w) + f(w_y - v_z, u_z - w_x, v_x - u_y) \\
&= (\nabla f) \times (u, v, w) + f(\nabla \times A)
\end{aligned}
$$

2 Problem 17 from Vector Calculus Notes

3 Problem 18 from Vector Calculus Notes

Let

$$A = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$$

and

$$B = 3y\mathbf{i} - 2x\mathbf{j}$$

(a) Check Product Rule 2

For the left hand side of rule 2, we have

$$
\begin{aligned}
\nabla(A \cdot B) &= \nabla(3xy - 4xy) \\
&= \nabla(-xy) \\
&= (-y, -x, 0)
\end{aligned}
$$

and the right hand side we have

$$
\begin{aligned}
A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A &= A \times (0, 0, -5) + B \times (0, 0, 0) + (A \cdot \nabla)B + (B \cdot \nabla)A \\
&= (-10y, 5x, 0) + (0, 0, 0) + (A \cdot \nabla)B + (B \cdot \nabla)A \\
&= (-10y, 5x, 0) + (A \cdot \nabla)B + (B \cdot \nabla)A \\
&= (-10y, 5x, 0) + \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right) B \\
&\quad + \left(3y\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}\right) A \\
&= (-10y, 5x, 0) + (6y, -2x, 0) + (3y, -4x, 0) \\
&= (-y, -x, 0)
\end{aligned}
$$

which is the same as the left hand side.

(b) Check Product Rule 4

For the left hand side of rule 4, we have

$$\begin{aligned}\nabla \cdot (A \times B) &= \nabla \cdot (6xz, 9yz, -2x^2 - 6y^2) \\ &= 6z + 9z + 0 \\ &= 3z\end{aligned}$$

and the right hand side we have

$$\begin{aligned}(\nabla \times A) \cdot B - A \cdot (\nabla \times B) &= (0, 0, 0) \cdot B - A \cdot (0, 0, -5) \\ &= -A \cdot (0, 0, -5) \\ &= -15z\end{aligned}$$

which is the same as the left hand side.

(c) Check Product Rule 6

For the left hand side of rule 4, we have

$$\begin{aligned}\nabla \times (A \times B) &= \nabla \times (6xz, 9yz, -2x^2 - 6y^2) \\ &= (-21y, 10x, 0)\end{aligned}$$

and the right hand side we have

$$\begin{aligned}(B \cdot \nabla)A - (A \cdot \nabla)B + A(\nabla \cdot B) - B(\nabla \cdot A) &= \left(3y\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}\right) A - \left(x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 3z\frac{\partial}{\partial z}\right) B \\ &\quad + A(\nabla \cdot B) - B(\nabla \cdot A) \\ &= (3y, -4x, 0) - (6y, -2x, 0) + A(\nabla \cdot B) - B(\nabla \cdot A) \\ &= (-3y, -2x, 0) + A(\nabla \cdot B) - B(\nabla \cdot A) \\ &= (-3y, -2x, 0) + A(0) - B(1 + 2 + 3) \\ &= (-3y, -2x, 0) - 6B \\ &= (-3y, -2x, 0) - (18y, -12x, 0) \\ &= (-21y, 10x, 0)\end{aligned}$$

which is the same as the left hand side.

4 Problem 19 from Vector Calculus Notes

5 Problem 20 from Vector Calculus Notes

6 Problem 21 from Vector Calculus Notes