Math 509: Advanced Analysis
Homework 10

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http://coursework.tylerlogic.com/courses/upenn/math509/homework10

1 Problem 22 from Vector Calculus Notes

2 Problem 23 from Vector Calculus Notes

3 Problem 24 from Vector Calculus Notes

According to the notes, an infinitesimal displacement has

$$dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = d\ell = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi} \tag{3.1}$$

which gives us a relationship between cartesian and spherical coordinates. Now we also know that

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

Differentiating each of these equations yields

$$\begin{aligned} dx &= \sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi \\ dy &= \sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned}$$

We can then plug these values into the left-hand side of equation 3.1 to obtain

$$\begin{aligned} dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi} = {} & (\sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi)\mathbf{i} + \\ & (\sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi)\mathbf{j} + \\ & (\cos\theta dr - r\sin\theta d\theta)\mathbf{k} \end{aligned}$$

Grouping the right-hand side by dr, $rd\theta$, and $r\sin\theta d\varphi$ we then obtain

$$\begin{aligned} dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\varphi\hat{\varphi} = {} & (\sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k})dr \\ & + (\cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k})rd\theta \\ & + (\cos\varphi\mathbf{j} - \sin\varphi\mathbf{i})r\sin\theta d\varphi \end{aligned}$$

implying that

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\varphi\mathbf{i} + \sin\theta\sin\varphi\mathbf{j} + \cos\theta\mathbf{k} \\ \hat{\theta} &= \cos\theta\cos\varphi\mathbf{i} + \cos\theta\sin\varphi\mathbf{j} - \sin\theta\mathbf{k} \\ \hat{\varphi} &= -\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j} \end{aligned}$$

4 Problem 25 from Vector Calculus Notes

5 Problem 26 from Vector Calculus Notes

6 Problem 27 from Vector Calculus Notes

7 Problem 28 from Vector Calculus Notes

8 Problem 29 from Vector Calculus Notes