Math 55a: Honors Abstract Algebra
Homework 2

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1 Exercise 7 On Page 35 Of Axler; From That, Deduce Exercises 5 And 6

Exercise seven's proof. Assume that we have a sequence of vectors, $v_1, v_2, \ldots$ such that $(v_1, v_2, \ldots, v_n)$ is linearly independent for all positive integers n. In Artin, we see that a finite set of vectors that spans V must be at least as large as any set of vectors that is linearly independent [1, pg 92]. Hence we can see that because we are gaurenteed to have an arbitrarily large, finite set of linearly independent vectors, then there exists no finite set of vectors that spans V, i.e. no LIST of vectors that spans V. Thus the space if infinite-dimensional.

For the reverse direction, if our vector space V is infinite-dimensional, then there exists no list of vectors that spans the space. Hence, we can find a linearly independent set of vectors in the space, call it L. We know they exist! The space is infinite-dimensional and any one non-zero vector will make a linearly independent set! Since we know that $span(L) \neq V$, then we can find a vector in V that is not in $span(L)$, call this v'. Thus the set $L \cup \{v'\}$ will also be linearly independent. Continuing this pattern allows us to create a linearly independent set of vectors of any size, i.e. there exists a sequence of vectors $v_1, v_2, \ldots$ such that $(v_1, v_2, \ldots, v_n)$ is linearly independent for all positive integers n.

Deduce exercise five's result from the above proof. To deduce five from the above result, we need to find an infinite sequence of linearly independent vectors. Hence we can take a page out of the standard basis' book with the following set of vectors.

$$
\begin{aligned}
v_1 &= (1, 0, 0, \ldots) \\
v_2 &= (0, 1, 0, 0, \ldots) \\
v_3 &= (0, 0, 1, 0, 0, \ldots) \\
&\vdots
\end{aligned}
$$

Certainly $(v_1, v_2, \ldots, v_n)$ will be linearly independent for all positive integers n, so therefore F^∞ is infinite-dimensional.

Deduce exercise six's result from the initial proof. Now we need to find another sequence of linearly independent vectors. Hence we go in with the "simple" mindset and can discover the following infinite sequence of linearly independent vectors.

$$
\begin{aligned}
v_1 &= x \\
v_2 &= x^2 \\
v_3 &= x^3 \\
&\vdots
\end{aligned}
$$

Hence we have that $\mathbb{R}^\mathbb{R}$ is infinite-dimensional.

2 Give the dimensions of the vector spaces generated by each of the following sets of vectors.

(a) The vectors $(1, i, 1+i, 0)$, $(-i, 1, 1-i, 0)$, and $(1-i, 1+i, 2, 0)$ in $\mathbb{C}^4$

By the following equation we have that the second vector is a multiple of the first vector.

$$-i(1, i, 1+i, 0) = (-i, 1, 1-i, 0)$$

Similarly, we can see that the third vector is a multiple of the first.

$$(1 - i)(1, i, 1 + i, 0) = (1 - i, 1 + i, 2, 0)$$

(b)   The functions $f(x) = \sin x$, $\sin(x + \pi/6)$, and $\sin(x + \pi/3)$ in $\mathbb{R}^{\mathbb{R}}$

By the following trigonometric identity [3]

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

we can obtain these two forms for $\sin(x + \pi/3)$ and $\sin(x + \pi/6)$ equations.

$$
\begin{aligned}
\sin(x + \pi/3) &= \sin x \cos(\pi/3) + \cos x \sin(\pi/3) \\
&= \frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x \\
\sin(x + \pi/6) &= \sin x \cos(\pi/6) + \cos x \sin(\pi/6) \\
&= \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x
\end{aligned}
$$

Hence, we can solve for $\cos x$ in both of these equations. By setting these two different solutions of $\cos x$ equal to each other, we have the following set of equations.

$$
\begin{aligned}
\frac{2}{\sqrt{3}} \sin(x + \pi/3) - \frac{1}{\sqrt{3}} \sin x &= 2 \sin(x + \pi/6) - \sqrt{3} \sin x \\
\sqrt{3} \sin x - \frac{1}{\sqrt{3}} \sin x &= 2 \sin(x + \pi/6) - \frac{2}{\sqrt{3}} \sin(x + \pi/3) \\
\sin x &= \sqrt{3} \sin(x + \pi/6) - \sin(x + \pi/3)
\end{aligned}
$$

And thus we can see that $\sin x$ is linear combination of the other two functions.

So at most, the dimension of the spanning set of this set of vectors is two. However, because $\sin(x + \pi/3)$ is not a scalar multiple of $\sin(x + \pi/6)$, we can conclude that the dimension of the span of this set of vectors can not be one either, leaving us with a dimension of two.

(c)   The n vectors given in the homework description (nope, not writing them out)

3   Give a basis for the specified subspaces.

(a)   The subspace $\{x^{\rightarrow} : x_1 + x_2 + \cdots + x_n = 0\}$ in F^n

Since all vectors are such that the sum of their elements must be zero, then we have that

$$x_1 = -x_2 - x_3 - \cdots - x_n$$

indicating that all vectors in the space have the following form.

$$(-x_2 - x_3 - \cdots - x_n, x_2, x_3, \ldots, x_n) \tag{3.1}$$

This gives us a nice way to derive a spanning set, and as well see, also an independent one. We can break down the vector in equation 3.1 as a sum of $n - 1$ vectors giving us a spanning set for the subspace. The break-down is as follows.

$$
\begin{aligned}
(-x_2 - x_3 - \cdots - x_n, x_2, x_3, \ldots, x_n) &= (-x_2, x_2, 0, \ldots, 0) + (-x_3, 0, x_3, 0, \ldots, 0) + \cdots + (-x_n, 0, \ldots, 0, x_n) \\
&= x_2(-1, 1, 0, \ldots, 0) + x_3(-1, 0, 1, 0, \ldots, 0) + \cdots + x_n(-1, 0, \ldots, 0, 1)
\end{aligned}
$$

We can also see that, due to equation 3.1, the set of vectors $(-1, 1, 0, \ldots, 0) + (-1, 0, 1, 0, \ldots, 0) + \cdots + (-1, 0, \ldots, 0, 1)$ is linearly independent.

(b)   The subspace $V = \{P : P(-3) = 0\}$ of $\mathcal{P}$

Immediately, the standard basis of $\mathcal{P}$ comes to mind. So we start our meddling there. We need each polynomial in the standard basis to be zero on input of -3, hence we account for that by subtracting $(-3)^n$ from each x^n in the standard basis to obtain the following set of vectors.

$$\{x^n - (-3)^n : n \in \mathbb{N}\} \tag{3.2}$$

Its easy to see that this set is linearly independent since no linear combination of any elements in the set will result in the necessary exponential of x to be equivalent to another element of the set.

Assume, by way of contradiction, that the set specified in 3.2 does not span V. Therefore there exists some polynomial, $p(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$, such that $p(-3) = 0$ but $p(x) \notin \text{span}(\{x^n - (-3)^n : n \in \mathbb{N}\})$. Thus we have that

$$a_0 + a_1(-3)^1 + a_2(-3)^2 + \cdots + a_n(-3)^n = 0$$

which in turn demands the following.

$$a_0 = -\sum_{i=1}^{n} a_i(-3)^i$$

Hence we have that

$$p(x) = a_0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n = -\sum_{i=1}^{n} a_i(-3)^i + \sum_{i=1}^{n} a_i x^i = \sum_{i=1}^{n} a_i \left(x^i - (-3)^i\right)$$

which implies that $p(x)$ is indeed in the span of the set in 3.2. This is the contradiction of our above assumption and therefore the set $\{x^n - (-3)^n : n \in \mathbb{N}\}$ is a basis for V.

4   Polynomial Interpolation

Let $a_1, a_2, \ldots, a_n$ be distinct elements of a field F.

Assume that for any n elements, $p_1, p_2, \ldots, p_n$ in F, there exists a unique polynomial $q \in \mathcal{P}(F)$ of degree less than n such that $q(a_i) = p_i$ for all $i = 1, 2, \ldots, n$. Letting $c_1, c_2, \ldots, c_n$ be the coefficients of q, then

$$Ac = p$$

where c is the vector of coefficients of q, $p = (p_1, p_2, \ldots, p_n)$, and A is an $n \times n$ matrix such that the ith row is the ith vector of the set $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ (taking rows to be indexed initially at 0). Since this UNIQUE equation is ALWAYS true for ALL possible vectors p in F^n, then the above assumption is akin to assuming that the rows of A are a basis for F^n, which certainly implies that the set of vectors $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ are linearly independent.

Conversely, assume that the n vectors $v_i := (a_1^i, a_2^i, \ldots, a_n^i)$ with $(0 \le i < n)$ in F^n are linearly independent. Therefore we have that the this set of vectors is also a basis for F^n, given the number of vectors. Hence EVERY vector $(p_1, p_2, \ldots, p_n)$ in F^n has a unique respresentation of the form

$$c_1(1, \ldots, 1) + c_2(a_1, \ldots, a_n) + \cdots + c_n(a_1^{n-1}, \ldots, a_n^{n-1}) = (c^T \alpha_1, c^T \alpha_2, \ldots, c^T \alpha_n)$$

where $c = (c_1, c_2, \ldots, c_n)$ and $\alpha_i = (1, a_i, a_i^2, \ldots, a_i^{n-1})$. Therefore if we construct a polynomial in $\mathcal{P}(F)$

$$q = c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}$$

then q would be a polynomial of degree less than n such that for all $p_1, p_2, \ldots, p_n \in F$ q is the unqiue polynomial such that $q(a_i) = p_i$ for all $i = 1, 2, \ldots, n$.

5 Complexification of a real vector space

Since only the scalar multiplication of V_C differs from the vector space $V \oplus V$, then we need only check that a complex scalar is distributive across the sum of two vectors (actually we need to check that $1v = v$ for all $v \in V_C$ but that's trivial). By the following sequence of equations, we get our desired distributive properties.

$$
\begin{aligned}
(a+bi)((v_1, w_1) + (v_2, w_2)) &= (a+bi)(v_1 + v_2, w_1 + w_2) \\
&= (a(v_1 + v_2) - b(w_1 + w_2), a(v_1 + v_2) + b(w_1 + w_2)) \\
&= ((av_1 - bw_1) + (av_2 - bw_2), (aw_1 + bv_1) + (aw_2 + bv_2)) \\
&= (av_1 - bw_1, aw_1 + bv_1) + (av_2 - bw_2, aw_2 + bv_2) \\
&= (a+bi)(v_1, w_1) + (a+bi)(v_2, w_2)
\end{aligned}
$$

If V is finite-dimensional, then V_C would have the same dimension as the $V \oplus V$ direct product, i.e. $2 \dim(V)$.

6 Dimension of K-vector spaces in F

Holding off completion until the chapter in Artin corresponding to this subject matter has been reached. Yes, this also means that despite not reading about the subject I was still unable to solve the problem.

7 The number of ordered bases

Let V be the F-vector space F^n where F is a finite field of size q.

(i)

Since elements of V are n-tuples made up of elements of F, then each "slot" of the tuple has q possible values, yielding that V has q^n distinct vectors.

As for the number of ordered bases, each such basis is made up of n linearly independent vectors since $V = F^n$ has dimension n. So we'll proceed by counting to find the number of lists (i.e order matters) of independent vectors of size n. The number of lists of linearly independent vectors of size one is simply the number of nonzero vectors, $q^n - 1$. To obtain a list of two linerarly independent vectors, we pick a second vector outside of the span of the first; note that this vector will be nonzero as the zero vector is contained within the span of any set of vectors. Since the span of the first vector has size q (i.e. the total number of linear combinations, since each linear combination is a unique vector since the vectors are linearly independent), then our number of choices for the second is $q^n - q$. This gives us $(q^n - 1)(q^n - q)$ lists of linearly independent vectors of size two. Since the span of two linearly independent vectors would have size q^2, our choices for a third vector are limitied to $q^n - q^2$ vectors giving us $(q^n - 1)(q^n - q)(q^n - q^2)$ linearly independent lists of size three. So finally, to add a kth linearly independent vector to a list of $k-1$ linearly independent vectors, we would have $q^n - q^{k-1}$ choices, yielding $(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})$ linearly independent lists of size k. Thus when $k = n$ the list will be a basis, and therefore the number of ordered bases is

$$
(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = \prod_{i=0}^{n-1} (q^n - q^i)
$$

(ii)

Each subspace of dimension k will be the span of some k linearly independent vectors of V. From the above proof we know that the number of lists of k linearly independent vectors is

$$
\prod_{i=0}^{k-1} (q^n - q^i)
$$

however, this counts different orderings of the k vectors, which matters not as different orderings of the same vectors results in identical subspaces. So we must divide out by the number of orderings of the k vectors. This leaves us with

$$\frac{1}{k!}\prod_{i=0}^{k-1}\left(q^n - q^i\right)$$

8   Torsion Subgroups

Let M be a module over $\mathbb{Z}$, $M[n]$ be the set of n-torsion elements, and $M_{\text{tors}} = \cup_{n\neq0}M[n]$ be the set of torsion elements of M.

(i)

$M[n]$ is a subgroup. Let a and b be n-torsion elements of M. Then we have that $n(a+b) = na+nb = 0+0 = 0$, and thus $M[n]$ is closed. Since $n0 = 0$ then $M[n]$ contains the identity element. Finally since $(n-1)a + a = (n-1+1)a = na = 0$, $a + (n-1)a = (1+n-1)a = na = a$, and $n((n-1)a) = (n-1)na = (n-1)0 = 0$, then each element of $M[n]$ has its inverse in $M[n]$.

M_{tors} is a subgroup. Letting $a, b \in M_{\text{tors}}$ we know that there exists integers n_a and n_b such that $n_a a = 0$ and $n_b b = 0$. Given this, we have $n_a n_b(a + b) = n_a n_b a + n_a n_b b = n_b n_a a + n_a 0 = n_b 0 = 0$, so M_{tors} is closed. Again, since any integer times the identity of M will result in the identity of M, then it is in M_{tors}. As proven in the previous paragraph, the inverse of a torsion element will be in the same n-torsion subgroup as the torsion element itself, which implies that M_{tors} is closed for inverses. Therefore M_{tors} is a subgroup of M.

(ii)

9   Divisible Groups/Modules

Let M be an abelian group.

(i)   Non-trivial Divisible Group

An example of an abelian group which only has zero as a torsion element is $\mathbb{Q}$. For if some $p/q \in \mathbb{Q}$ with nonzero integers p and q had that $np/q = 0$ for some nonzero n. Then $np = 0$, but this can't happen as n and p are both nonzero integers, and therefore not zero divisors.

(ii)   Torsion Group

An example of a torsion group would be $\mathbb{Z}/n\mathbb{Z}$ for any n since adding any element to itself n times would result in the zero element. Thus every element is a torsion element, and therefore $\mathbb{Z}/n\mathbb{Z}$ would be a torsion group.

References

[1] Artin, Michael. *Algebra*. Prentice Hall. Upper Saddle River NJ: 1991.

[2] Axler, Sheldon. *Linear Algebra Done Right* 2nd Ed. Springer. New York NY: 1997.

[3] Valdex-Sanchez, Luis. *Table of Trigonometric Identities* Last Edited: December 3, 1996. Available at: http://www.sosmath.com/trig/Trig5/trig5/trig5.html