Math 312: Linear Algebra
Homework 1

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1

For this problem, let us assume that $a, b, c, d, e, f \in \mathbb{F}$ for some field $\mathbb{F}$. The system of equations

$$\begin{aligned} ax + cy &= e \\ bx + dy &= f \end{aligned}$$

has the matrix form of $Au = v$ where

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \; u = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ and } v = \begin{pmatrix} e \\ f \end{pmatrix}$$

or alternatively

$$x \begin{pmatrix} a \\ b \end{pmatrix} + y \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

Given this form, we see that in order for a solution to exist for any $e, f \in \mathbb{F}$ then a, b, c, d need to be such that the vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \begin{pmatrix} c \\ d \end{pmatrix}$$

spans the entire space $\mathbb{F}^2_{\text{cols}}$, where $\mathbb{F}^2_{\text{cols}}$ denotes the vector space of two dimensional column-vectors.

2

Let $v_1, \ldots, v_n \in \mathbb{F}^n$ for $\mathbb{F}^n$ over $\mathbb{F}$ and $A \in \mathrm{M}_{n\times m}(\mathbb{F})$ such that $v_i = (a_{1i}, \ldots, a_{ni})$ with each $a_{ij} \in \mathbb{F}$ and $A_{ij} = a_{ij}$.
Given this definition of A, we see that

$$A = (v_1^t | v_2^t | \cdots | v_m^t)$$

that is, $v_1^t, \ldots, v_n^t$ are the columns of A.

(a)

Let $\mathrm{Span}(v_1, \ldots, v_m) = \mathbb{F}^n$. Then for any $b \in \mathbb{F}^n$ there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that $c_1 v_1 + \cdots + c_m v_m = b$. Since $v_1, \ldots, v_m, b$ are all row vectors, this is akin to saying that

$$c_1 v_1^t + \cdots + c_m v_m^t = b^t$$

but this is just $Ac^t = b^t$ where $c^t = (c_1, \ldots, c_m)^t$. Hence for any vector b^t in the column-space of $\mathbb{F}^n$, $Ax = b^t$ will have a solution, i.e. the row-reduced echelon form of A will have a pivot in each row, so it will have n pivots.

(b)

Let the row-reduced echelon form of A have n pivots, that is, every row of A has a pivot since it has n rows. This means that no matter the $b \in \mathbb{F}^n_{\text{cols}}$, $Ax = b$ will have a solution. In other words, for any b there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that $x_1 v_1^t + \cdots + x_m v_m^t = b$. This is equivalent to saying that for any $b' \in \mathbb{F}^n$ there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that

$$c_1 v_1 + \cdots + c_m v_m = b'$$

Hence $v_1, \ldots, v_m$ span $\mathbb{F}^n$.

(c)

Given our solution in part (a) if $\mathrm{Span}(v_1, \ldots, v_m) = \mathbb{F}^n$, then the matrix A with the i^{th} column being v_i, will have n pivots in its RREF. Because A is an $n \times m$ matrix and because each pivot in a RREF matrix must be in a different row and column than any other pivot, then $m \geq n$.

3

For the following solutions let $\mathbb{F}$ be a field and fix $A \in \mathrm{M}_{m\times n}(\mathbb{F})$.

(a)

Let S be the subset of $\mathrm{M}_{m\times k}(\mathbb{F})$ defined by $\{A \cdot B : B \in \mathrm{M}_{n\times k}(\mathbb{F})\}$. For $b, b' \in \mathbb{F}$ and $C, C' \in S$ there exist $B, B' \in \mathrm{M}_{m\times k}(\mathbb{F})$ such that both $C = AB$ and $C' = AB'$. Making use of this and Lemma A.1 we attain

$$bC + b'C' = b(AB) + b'(AB') = A(bB) + A(b'B') = A(bB + b'B')$$

and thus $bC + b'C' \in S$. Given this and that $0_{\mathrm{M}_{m\times k}(\mathbb{F})}$ are both in S, so S is a subspace of $\mathrm{M}_{n\times k}(\mathbb{F})$.

(b)

Let S be the subset of $\mathrm{M}_{k\times n}(\mathbb{F})$ defined by $\{B \cdot A : B \in \mathrm{M}_{k\times m}(\mathbb{F})\}$. For $b, b' \in \mathbb{F}$ and $C, C' \in S$ there exists $B, B' \in \mathrm{M}_{k\times m}(\mathbb{F})$ such that $C = BA$ and $C' = B'A$. Making use of this we obtain the following.

$$bC + b'C' = b(BA) + b'(B'A) = (bB)A + (b'B')A = (bB + b'B')A$$

Therefore $bC + b'C' \in S$. Since this is true and $0_{\mathrm{M}_{k\times n}(\mathbb{F})} \in S$, then S is a subspace of $\mathrm{M}_{k\times n}(\mathbb{F})$.

(c)

Let S be the subset of $\mathrm{M}_{n\times k}(\mathbb{F})$ defined by $\{B \in M_{n\times k}(\mathbb{F}) : A \cdot B = 0\}$. For $b, b' \in \mathbb{F}$ and $B, B' \in S$ we have

$$A(bB + b'B') = A(bB) + A(b'B') = b(AB) + b'(AB') = b0 + b'0 = 0$$

since $AB = 0$ and $AB' = 0$. This implies $bB + b'B' \in S$, so S is a subspace of $\mathrm{M}_{n\times k}(\mathbb{F})$.

(d)

Let S be the subset of $\mathrm{M}_{k\times m}(\mathbb{F})$ defined by $\{B \in M_{k\times m}(\mathbb{F}) : B \cdot A = 0\}$. For $b, b' \in \mathbb{F}$ and $B, B' \in S$ we have

$$(bB + b'B')A = (bB)A + (b'B')A = b(BA) + b'(B'A) = b0 + b'0 = 0$$

by Lemma A.1 and because $BA = 0$ and $B'A = 0$. This indicates that $bB + b'B' \in S$. Thus S is a subspace of $M_{k\times m}(\mathbb{F})$

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(a)

This subset W_1 is a subspace of $\mathbb{R}^3$ since for $a, b \in \mathbb{R}$ and $(3x, x, -x), (3y, y, -y) \in \mathbb{R}^3$ we have

$$a(3x, x, -x) + b(3y, y, -y) = (3ax, ax, -ax) + (3by, by, -by) = (3(ax + by), ax + by, -(ax + by))$$

which is an element of W_1.

(b)

This subset W_2 is not a subspace of $\mathbb{R}^3$ because it does not contain the zero vector since the zero vector does not fit the form of vectors in W_2, i.e. there are no $a_2, a_3\mathbb{R}$ such that

$$(0,0,0) = (a_3 + 2, a_2, a_3)$$

(c)

Given $(x,y,z),(x',y',z') \in W_3$ we have that

$$2x - 7y + z = 0 \text{ and } 2x' - 7y' + z' = 0$$

So for $a, b \in \mathbb{F}$ we have the following, given that $a(x,y,z) + b(x',y',z') = (ax + bx', ay + by', az + bz')$

$$2(ax+bx') - 7(ay+by') + (az+bz') = (2ax - 7ay + az) + (2bx' - 7by' + bz') = a(2x - 7y + z) + b(2x' - 7y' + z') = 0$$

then $a(x,y,z) + b(x',y',z') \in W_3$, and thus W_3 a subspace of $\mathbb{R}^3$.

(d)

Given $(x,y,z),(x',y',z') \in W_4$ we have that

$$x - 4y - z = 0 \text{ and } x' - 4y' - z' = 0$$

So for $a, b \in \mathbb{F}$ we have the following, given that $a(x,y,z) + b(x',y',z') = (ax + bx', ay + by', az + bz')$

$$(ax + bx') - 4(ay + by') - (az + bz') = (ax - 4ay - az) + (bx' - 4by' - bz') = a(x - 4y - z) + b(x' - 4y' - z') = 0$$

then $a(x,y,z) + b(x',y',z') \in W_4$, and hence W_4 is a subspace of $\mathbb{R}^3$

(e)

This subset, W_5, is not a subspace of $\mathbb{R}^3$ as it does not contain the zero vector since $0 + 2(0) - 3(0) = 0 \neq 1$.

(f)

The subset W_6 requires that all elements, (x,y,z), in it have that $5x^2 - 3y^2 + 6z^2 = 0$, that is to say that

$$y = \pm\sqrt{5/3x^2 + 2z^2}$$

So $(3, \sqrt{17}, 1)$ and $(3, \sqrt{23}, 2)$ are both in W_6. They're sum, however, is $(6, \sqrt{17} + \sqrt{23}, 3)$ which since

$$5(6^2) - 3(\sqrt{17} + \sqrt{23})^2 + 6(3^2) = 180 - 3(17 + 2\sqrt{17(23)} + 23) + 54 = 234 - (51 + 6\sqrt{17(23)} + 69) = 252 + 6\sqrt{17(23)} \neq 0$$

then W_6 is not closed under addition, and so its not a subspace of $\mathbb{R}^3$.

5

(a)

Let S be the set of all matrices in $A \in \mathrm{M}_{m\times n}(\mathbb{F})$ for which the entries of A are all zero except for any one entry. There will only be mn of these matrices in $\mathrm{M}_{m\times n}(\mathbb{F})$, and they will span the space.

More formally, we can define S as the set $\{A^k_\ell : A^k_\ell \in \mathrm{M}_{m\times n}(\mathbb{F}), k \in \{1,\dots,m\}, \ell \in \{1,\dots,n\}\}$ where each entry of the matrix A^k_ℓ is defined by

$$(A^k_\ell)_{ij} = \delta_k(i)\delta_\ell(j)$$

where δ is the Kronecker-delta according to the field $\mathbb{F}$, i.e. we use the field's multiplicative and additive identities rather than the integers 1 and 0.

With this more specific definition, we can write any matrix $B \in M_{m\times n}(\mathbb{F})$ with the following linear combination of elements of S.

$$B = \sum_{i=1}^{m} \left(B_{i1}A^i_1 + B_{i2}A^i_2 + \cdots + B_{in}A^i_n\right)$$

(b)

Here we can use our ideas from the previous problem. We'll continue our trend of using 1 and 0 in order to be able to generate the whole space. We define $S \subseteq \mathrm{M}_{n\times n}(\mathbb{F})$ in much the same manner as we did before, as the set $\{A^k_\ell : A^k_\ell \in \mathrm{M}_{n\times n}(\mathbb{F}), k \in \{1,\dots,n\}, \ell \in \{1,\dots,n\}, k \leq \ell\}$, however, we need to redefine A^k_ℓ to get the symmetry we desire. Putting it to $\delta_k(i)\delta_\ell(j) + \delta_k(j)\delta_\ell(i)$ gets us our symmetry, but has the annoying side effect of resulting in a value of one being added to itself whenever $\ell = k$. Hence, we need to "neglect" one of the terms whenever ℓ equals k. To do this, simply multiply one term by $(1 - \delta_k(\ell))$[1] to get the following entry-wise definition.

$$(A^k_\ell)_{ij} = (1 - \delta_k(\ell))\delta_k(i)\delta_\ell(j) + \delta_k(j)\delta_\ell(i)$$

With this we can see that when $k = \ell$,

$$(A^k_\ell)_{ij} = \delta_k(j)\delta_\ell(i) = \delta_\ell(j)\delta_k(i) = \delta_k(i)\delta_\ell(j) = (A^k_\ell)_{ji}$$

and when $k \neq \ell$,

$$(A^k_\ell)_{ij} = \delta_k(i)\delta_\ell(j) + \delta_k(j)\delta_\ell(i) = \delta_k(j)\delta_\ell(i) + \delta_k(i)\delta_\ell(j) = (A^k_\ell)_{ji}$$

so we confirm the symmetry of each matrix in S.

Since the number of elements of S is governed by the possible values for k and ℓ, then we look to those for the size of S. The value of k ranges from 1 to n and ℓ will always be greater than or equal to k, meaning that when $k = 1, k = 2, \dots, k = n$ the are $n, n-1, \dots, 1$ values of ℓ. Summing these we get

$$n + (n-1) + \cdots + 1 = \left(\frac{n}{2}\right)(n+1) = \frac{n(n+1)}{2}$$

This is the size of S, as needed.

Now, any $B \in \mathrm{Sym}_n$ can be written as a linear combination of the elements of S in the following manner.

$$B = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} B_{ij}A^i_j\right)$$

[1] Notice that this is the opposite of $\delta_k(\ell)$, being one whenever it's zero and zero whenever it's one.

(c)

Let $S \subseteq \mathrm{M}_{n\times n}(\mathbb{F})$ be the set $\{A^k_\ell : A^k_\ell \in \mathrm{M}_{n\times n}(\mathbb{F}), k \in \{1,\ldots,n\}, \ell \in \{1,\ldots,n\}, k < \ell\}$, but define A^k_ℓ, entry-wise, in the following way.

$$(A^k_\ell)_{ij} = \delta_k(i)\delta_\ell(j) - \delta_k(j)\delta_\ell(i)$$

We can see this results in skew-symmetry,

$$(A^k_\ell)_{ij} = \delta_k(i)\delta_\ell(j) - \delta_k(j)\delta_\ell(i) = -(\delta_k(j)\delta_\ell(i) - \delta_k(i)\delta_\ell(j)) = -(A^k_\ell)_{ji}$$

as well as for when $i = j$ (the diagonals)

$$(A^k_\ell)_{ij} = \delta_k(i)\delta_\ell(j) - \delta_k(j)\delta_\ell(i) = \delta_k(i)\delta_\ell(i) - \delta_k(i)\delta_\ell(i) = 0$$

which is required in Skew_n.

Given that k ranges over $1,\ldots,n$, $k < \ell$, and ℓ ranges over $k+1,\ldots,n$ then the size of S is the following.

$$(n-1) + (n-2) + \cdots + 1 = n\left(\frac{n-1}{2}\right) = \frac{n(n-1)}{2}$$

Now, any $B \in \mathrm{Skew}_n$ can be written as a linear combination of the elements of S in the following manner.

$$B = \sum_{i=1}^{n}\left(\sum_{j=i+1}^{n} B_{ij}A^i_j\right)$$

6

Let V be a vector space over a field $\mathbb{F}$ and there exist some subspaces W_1 and W_2.

(a) $W_1 \cap W_2$

Let u and v each be vectors in $W_1 \cap W_2$ and $a, b \in \mathbb{F}$. Then u and v are also each individually in W_1 and W_2, but being subspaces of V, W_1 and W_2 then both also contain the sum $au + bv$ and hence $au + bv \in W_1 \cap W_2$.

(b) $W_1 \cup W_2$

It is not always true that the union of two subspaces is a subspace. To derive a counter-example, we can apply what we learned about some subspaces of $\mathrm{M}_{m\times n}(\mathbb{F})$ in problem 3.

Letting A_1 and A_2 be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

respectively, we know that $W_1 = \{A_1B : B \in M_{2\times 2}(\mathbb{F})\}$ and $W_2 = \{A_2B : B \in M_{2\times 2}(\mathbb{F})\}$ are subspaces of $M_{2\times 2}(\mathbb{F})$. Elements of W_1 have thus form

$$A_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \tag{6.1}$$

and likewise elements of W_2 are

$$A_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \tag{6.2}$$

Since the identity matrix is in $M_{2\times 2}(\mathbb{F})$ then $A_1 \in W_1$ and $A_2 \in W_2$, and therefore they are both in $W_1 \cup W_2$. If this union were a subspace, then

$$A_1 + A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

would need to be in it, but this is neither in the form of equation 6.1 nor equation 6.2. So $W_1 \cup W_2$ is not a subspace.

(c) $W_1 + W_2$

Let $u, v \in W_1 + W_2$. Hence there exist vectors $w_{1u}, w_{1v} \in W_1$ and $w_{2u}, w_{2v} \in W_2$ such that $u = w_{1u} + w_{2u}$ and $v = w_{1v} + w_{2v}$. By this we have that for $a, b \in \mathbb{F}$

$$au + bv = a(w_{1u} + w_{2u}) + b(w_{1v} + w_{2v}) = aw_{1u} + aw_{2u} + bw_{1v} + bw_{2v} = (aw_{1u} + bw_{1v}) + (aw_{2u} + bw_{2v})$$

and therefore $au + bv \in W_1 + W_2$. Thus $W_1 + W_2$ is a subspace of V.

7

(a)

A matrix $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ that is both symmetric and anti-symmetric must maintain the condition $A_{ij} = A_{ji} = -A_{ji}$. Only the zero matrix of $\mathrm{M}_{n\times n}(\mathbb{R})$ does this, so $\mathrm{Sym}_n \cap \mathrm{Skew}_n = \{0\}$.

Let $A \in \mathrm{M}_{n\times n}(\mathbb{R})$ and define matrices B and C by the following.

$$B = \frac{1}{2}(A + A^t) \text{ and } C = \frac{1}{2}(A - A^t)$$

With these definitions, $B, C \in \mathrm{M}_{n\times n}(\mathbb{R})$. We then also have that

$$B_{ij} = \left(\frac{1}{2}(A + A^t)\right)_{ij} = \frac{1}{2}(A_{ij} + (A^t)_{ij}) = \frac{1}{2}(A_{ij} + A_{ji}) = \frac{1}{2}((A^t)_{ji} + A_{ji}) = \frac{1}{2}(A_{ji} + (A^t)_{ji}) = \left(\frac{1}{2}(A + A^t)\right)_{ji} = B_{ji}$$

and

$$C_{ij} = \left(\frac{1}{2}(A - A^t)\right)_{ij} = \frac{1}{2}(A_{ij} - (A^t)_{ij}) = \frac{1}{2}(A_{ij} - A_{ji}) = -\frac{1}{2}(A_{ji} - A_{ij}) = -\frac{1}{2}(A_{ji} - (A^t)_{ji}) = -\left(\frac{1}{2}(A - A^t)\right)_{ji} = -C_{ji}$$

which indicate that $B \in \mathrm{Sym}_n$ and $C \in \mathrm{Skew}_n$. Looking at the sum of these two matrices

$$B + C = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = \frac{1}{2}(A + A^t + A - A^t) = \frac{1}{2}(2A) = A$$

we see that it's simply A, implying that $A \in \mathrm{Sym}_n \oplus \mathrm{Skew}_n$.

With the above two results, we have that $\mathrm{M}_{n\times n}(\mathbb{R}) = \mathrm{Sym}_n \oplus \mathrm{Skew}_n$.

(b)

Let both $f \in \mathfrak{O}$ and $f \in \mathfrak{E}$. Then for every $x \in \mathbb{R}$, $f(-x) = -f(x) = f(x)$, and thus f must be the constant zero function. This then implies that $\mathfrak{O} \cap \mathfrak{E} = \{f\} = \{0\}$ since the constant zero function is the zero vector of $C^0(\mathbb{R}, \mathbb{R})$.

Now let $h \in C^0(\mathbb{R}, \mathbb{R})$ and define the functions f and g by the following.

$$f(x) = \frac{1}{2}(h(x) + h(-x)) \text{ and } g(x) = \frac{1}{2}(h(x) - h(-x))$$

With these definitions, $f, g \in C^0(\mathbb{R}, \mathbb{R})$. Given these we see that

$$f(-x) = \frac{1}{2}(h(-x) + h(-(-x))) = \frac{1}{2}(h(-x) + h(x)) = \frac{1}{2}(h(x) + h(-x)) = f(x)$$

and

$$g(-x) = \frac{1}{2}(h(-x) - h(-(-x))) = \frac{1}{2}(h(-x) - h(x)) = -\frac{1}{2}(h(x) - h(-x)) = -g(x)$$

which informs us that $f \in \mathfrak{E}$ and $g \in \mathfrak{O}$. Looking at the sum of f and g,

$$(f+g)(x) = f(x)+g(x) = \frac{1}{2}(h(x)+h(-x))+\frac{1}{2}(h(x)-h(-x)) = \frac{1}{2}(h(x)+h(-x)+h(x)-h(-x)) = \frac{1}{2}(2h(x)+h(x)) = h(x)$$

we see that it is simply h. So $h \in \mathfrak{O} \oplus \mathfrak{E}$.

By the results of the two paragraphs above, we have that $C^0(\mathbb{R}, \mathbb{R}) = \mathfrak{O} \oplus \mathfrak{E}$

A Extraneous Proofs

While there are some things in the solutions above that appear obvious to me, there are others that did not. For the latter, these are some of the proofs I generated in order to convince myself of certain properties.

Lemma A.1 (Commuting Scalars Among Matrices) *Given a scalar $c \in \mathbb{F}$, $A \in M_{m\times\ell}(\mathbb{F})$, and $B \in M_{\ell\times n}(\mathbb{F})$ for some field $\mathbb{F}$, c can commute about the matrix multiplication of A and B, that is*

$$c(AB) = A(cB) \tag{A.3}$$

Proof. In this case, it suffices to show that the ij^{th} entry on the left-hand-side of A.3 is equivalent to the right-hand-side. Given this we have the following due to the multiplicative commutativity of $\mathbb{F}$.

$$(c(AB))_{ij} = c(AB)_{ij} = c\left(\sum_{k=1}^{\ell} A_{ik}B_{kj}\right) = \sum_{k=1}^{\ell} A_{ik}cB_{kj} = \sum_{k=1}^{\ell} A_{ik}(cB)_{kj} = (A(cB))_{ij} \qquad \square$$

References

[1] Friedberg, S.H. and Insel, A.J. and Spence, L.E. *Linear Algebra* 4e. Upper Saddle River: Pearson Education, 2003.