Math 312: Linear Algebra
Homework 2

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July 26, 2012
http://coursework.tylerlogic.com/math312/homework02

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(a)

Let $a, b \in \mathbb{F}$ and $A, B \in \mathrm{M}_{n\times n}(\mathbb{F})$. Then we have

$$\begin{aligned}\operatorname{tr}(aA+bB) &= \sum_{i=1}^{n}(aA+bB)_{ii}\\ &= \sum_{i=1}^{n}(aA_{ii}+bB_{ii})\\ &= a\sum_{i=1}^{n}A_{ii}+b\sum_{i=1}^{n}B_{ii}\\ &= a\operatorname{tr}(A)+b\operatorname{tr}(B)\end{aligned}$$

which gives us the linearity of trace.

(b)

Let $w, w' \in W$. Then the trace of w and w' are each zero. So for $a, b \in \mathbb{F}$ this yields the following by the linearity of trace.

$$\operatorname{tr}(aw+bw') = a\operatorname{tr}(w)+b\operatorname{tr}(w') = a(0)+b(0) = 0$$

So $aw+bw' \in W$, and thus W is a subspace.

(c)

Matrices in $\mathrm{M}_{n\times n}(\mathbb{F})$ will have the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $a, b, c, d \in \mathbb{F}$. Matrices in W will have the above form with the restriction that $a+d=0$, i.e. $d=-a$, so actually, all elements of W will have the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

With this form, one basis (maybe call it the standard basis for W) would be

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$$

since these matrices are linearly independent and

$$a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}+c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

With this, W has dimension three.

2 Problems from §2.4

(a) Problem 1

For this problem, we'll use the more intuitive subscript notation of the matrix representation of a linear operator from class and of Treil [2], rather than Friedberg, Insel, and Spence's notation.

— (a) —

Generally false, since inverses are unique and the inverse of $[T]_{\beta\alpha}$ is $[T^{-1}]_{\alpha\beta}$ by Theorem 2.8 [1, p. 101]. This could potentially be true if V, W, α, β, and T were chosen appropriately, say $V = W$ and $\alpha = \beta$.

— (b) —

True.

— (c) —

True since $[T]_{\beta\alpha}[v]_\alpha = [T(v)]_\beta$ for $v \in V$.

— (d) —

False. The space $\mathrm{M}_{2\times 3}(F)$ has dimension 6 and $\mathbb{F}^5$ has dimension 5.

— (e) —

True. Both $P_n(\mathbb{F})$ and $P_m(\mathbb{F})$ are over the same field and have the same dimension if and only $n = m$, so by Theorem 2.19 [1, p. 103], they are isomorphic.

— (f) —

False. The product of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

will be the two-by-two identity matrix, but neither are invertible as they are not square matrices.

— (g) —

True since A^{-1} satisfies the requirements of being an inverse of $(A^{-1})^{-1}$,

$$A^{-1}(A^{-1})^{-1} = (A^{-1}A)^{-1} = I \text{ and } (A^{-1})^{-1}A^{-1} = (AA^{-1})^{-1} = I$$

and inverses are unique [1, p. 100].

— (h) —

True by corollary two of theorem 2.18 [1, p. 102].

— (i) —

True. If A were an $m \times n$ matrix where $m \neq n$, then there would not exist any matrix B such that $AB = BA$ since AB and BA would have different dimensions.

(b) Problem 2

— (a) —

The linear transformation T is not invertible. The vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ have dimension 2 and 3, respectively, so they are not isomophic to each other. Because they are not isomorphic, no linear transformation between them can be invertible; more specifically T cannot.

— (b) —

The reasoning of the previous problem stands here as well.

— (c) —

This linear transformation is invertible.

Let $(a, b, c), (a', b', c') \in \mathbb{R}^3$ be such that $T(a, b, c) = T(a', b', c')$. Then $(3a-2c, b, 3a+4b) = (3a'-2c', b', 3a'+4b')$, which requires that $b = b'$. Due to this, $3a + 4b' = 4a' + 4b'$ implies that $a = a'$ which in turn tells us that $3a' - 2c = 3a' - 2c'$, and thus $c = c'$. So T is one-to-one.

Let $(x, y, z) \in \mathbb{R}^3$. Seeing that for $v = \left(\frac{z-4y}{3}, y, \frac{1}{2}(-x + z - 4y)\right)$

$$T(v) = \left(3\left(\frac{z-4y}{3}\right) - 2\left(\frac{1}{2}(-x+z-4y)\right), y, 3\left(\frac{z-4y}{3}\right) + 4y\right) = (z - 4y + x - z + 4y, y, z - 4y + 4y) = (x, y, z)$$

we have that T is onto. Because T is one-to-one and onto, then it is invertible.

— (d) —

$P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ differ in dimension, so as before, there is no invertible linear transformation between them.

— (e) —

$\mathrm{M}_{2\times 2}(\mathbb{R})$ and $P_2(\mathbb{R})$ differ in dimension, so as before, there is no invertible linear transformation between them.

— (f) —

This is an invertible linear transformation. Define $U : \mathrm{M}_{2\times 2}(\mathbb{R}) \to \mathrm{M}_{2\times 2}(\mathbb{R})$ by the following.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix}$$

With this definition we see

$$U\left(T\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = U\begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} a & (a+b)-a \\ c & (c+d)-c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$T\left(U\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = T\begin{pmatrix} b & a-b \\ c & d-c \end{pmatrix} = \begin{pmatrix} b+(a-b) & b \\ c & c+(d-c) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so $U \circ T = \mathbb{I}_V$ and $T \circ U = \mathbb{I}_W$. Thus T is invertible.

(c) Problem 3

— (a) —

The vector spaces $\mathbb{F}^3$ and $P_3(\mathbb{F})$ are not isomorphic as they do not have the same dimension. The former has dimension 3 and the latter, 4.

— (b) —

The vector spaces $\mathbb{F}^4$ and $P_3(\mathbb{F})$ are isomorphic since they have the same dimension and are both over $\mathbb{F}$.

— (c) —

The vector spaces $\mathrm{M}_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{F})$ are isomorphic since they have the same dimension and are both vector spaces over $\mathbb{F}$.

— (d) —

The vector space V has a dimension of three, as seen in problem 1, but $\mathbb{R}^4$ has dimension four, so these two vector spaces are not isomorphic.

(d) Problem 4

Let A and B be $n \times n$ invertible matrices. Then $B^{-1}A^{-1}$ is square, and we see both

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

and

$$(AB)(B^{-1}A^{-1}) = A^{-1}(B^{-1}B)A = A^{-1}IA = A^{-1}A = I$$

This implies that $B^{-1}A^{-1}$ is the inverse of AB. Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

(e) Problem 5

Let A be an invertible matrix. Then A is square and thus so is its transpose. Seeing that

$$(A^{-1})^t A^t = (AA^{-1})^t = I^t = I$$

and

$$A^t(A^{-1})^t = (A^{-1}A)^t = I^t = I$$

we know then that A^t is invertible and furthermore that its inverse is $(A^{-1})^t$.

(f) Problem 6

Let A be an invertible matrix and B some matrix such that $AB = 0$. Then we have the following sequence of equations.

$$\begin{aligned} AB &= 0 \\ A^{-1}(AB) &= A^{-1}0 \\ (A^{-1}A)B &= 0 \\ IB &= 0 \\ B &= 0 \end{aligned}$$

(g) Problem 7

Let A be an $n \times n$ matrix.

— (a) —

Let $A^2 = 0$. Assume for later contradiction that A is invertible. Then we have that $AA^{-1} = I$, which implies that

$$\begin{aligned} A(AA^{-1})A^{-1} &= AIA^{-1} \\ (AA)A^{-1}A^{-1} &= AA^{-1} \\ (0)A^{-1}A^{-1} &= I \\ 0 &= I \end{aligned}$$

however, the identity and zero matrices are not equal; we have a contradiction. Thus A is not invertible.

— (b) —

If $AB = 0$ for some nonzero $n \times n$ matrix B then A cannot be invertible as being so would contradict our result of our proof for the previous problem.

(h) Problem 9

Let A and B be $n \times n$ matrices such that their product AB is invertible. Then L_{AB} must be invertible as well. From this results

$$L_A L_{B(AB)^{-1}} = L_{AB} L_{(AB)^{-1}} = \mathbb{I}$$

and

$$L_{(AB)^{-1}A} L_B = L_{(AB)^{-1}} L_{AB} = \mathbb{I}$$

which informs us, respectively, that L_A is onto and L_B is $1-1$. But since these two maps each have domain and codomain of $\mathrm{M}_{n\times 1}(\mathbb{F})$, namely a domain and codomain of equal dimension, then Theorem 2.5 [1, p. 71] gives us that L_A and L_B are bijections, that is they are invertible. Hence A and B are invertible too.

(i) Problem 14

Let's look at the transformation $T : V \to \mathbb{F}^3$ defined by

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} cc \longmapsto (x, y - x, z)$$

which was derived in an attempt to "undo" the definition of V to get back into the space $\mathbb{F}^3$. Let $a, b \in \mathbb{F}$ and

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} cc \text{ and } \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} cc$$

be elements of V. By the following, we can see that T indeed belongs to $\mathcal{L}(V, \mathbb{F}^3)$.

$$\begin{aligned} T\left(a\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + b\begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix}\right) &= T\begin{pmatrix} ax+bx' & ay+by' \\ 0 & az+bz' \end{pmatrix} \\ &= (ax+bx', ay+by' - (ax+bx'), az+bz') \\ &= (ax+bx', a(y-x)+b(y'-x'), az+bz') \\ &= (ax, a(y-x), az) + (bx', b(y'-x'), bz') \\ &= a(x, y-x, z) + b(x', y'-x', z') \\ &= aT\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} + bT\begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} \end{aligned}$$

Now let's define the map $U : \mathbb{F}^3 \to V$ by

$$(a, b, c) \longmapsto \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix}$$

With this definition, we have that for $(a,b,c) \in \mathbb{F}^3$

$$T(U(a,b,c)) = T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a,(a+b)-a,c) = (a,b,c)$$

and for $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in V$

$$U\left(T\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = U(x, y-x, z) = \begin{pmatrix} x & x+(y-x) \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$$

Hence $T \circ U = \mathbb{I}_{\mathbb{F}^3}$ and $U \circ T = \mathbb{I}_V$, so T is an isomorphism from V to $\mathbb{F}^3$.

(j) Problem 16

Let B be an $n \times n$ invertible matrix and define $\Phi : \mathrm{M}_{n\times n}(\mathbb{F}) \to \mathrm{M}_{n\times n}(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Letting $a, a' \in \mathbb{F}$ we see the following for $A, A' \in \mathrm{M}_{n\times n}(\mathbb{F})$.

$$\Phi(aA + a'A') = B^{-1}(aA + a'A')B = (aB^{-1}A + a'B^{-1}A')B = aB^{-1}AB + a'B^{-1}A'B = a\Phi(A) + a'\Phi(A')$$

So Φ is linear. Define $\Phi' : \mathrm{M}_{n\times n}(\mathbb{F}) \to \mathrm{M}_{n\times n}(\mathbb{F})$ by $\Phi'(A) = BAB^{-1}$. Given this definition, we see

$$\Phi(\Phi'(A)) = \Phi\left(BAB^{-1}\right) = B^{-1}\left(BAB^{-1}\right)B = \left(B^{-1}B\right)A\left(B^{-1}B\right) = IAI = A$$

and

$$\Phi'(\Phi(A)) = \Phi'\left(B^{-1}AB\right) = B\left(B^{-1}AB\right)B^{-1} = \left(BB^{-1}\right)A\left(BB^{-1}\right) = IAI = A$$

which together with the linearity of Φ yields to us that Φ is invertible.

(k) Problem 22

Let $c_0, \ldots, c_n$ be distinct scalars from an infinite field $\mathbb{F}$. Define the map $T : P_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ by $T(f) = (f(c_0), \ldots, f(c_n))$. With this definition, we see that T is linear since for $a, b \in \mathbb{F}$ and $f, g \in P_n(\mathbb{F})$

$$T(af+bg) = ((af+bg)(c_0), \ldots, (af+bg)(c_n)) = a(f(c_0), \ldots, f(c_n)) + b(g(c_0), \ldots, g(c_n)) = aT(f) + bT(g)$$

Remembering from the first chapter of our book, [1], that a set of Lagrange polynomials in $P_n(\mathbb{F})$ is a basis, we, in an attempt to find an inverse for T, define $U : \mathbb{F}^{n+1} \to P_n(\mathbb{F})$ using the Lagrange polynomials $g_0, \ldots, g_n$ corresponding to $c_0, \ldots, c_n$ by $U(a_0, \ldots, a_n) = a_0g_0 + \cdots + a_ng_n$. With this definition, we have that

$$\begin{aligned} U(T(f)) &= U(T(a_0g_0 + \cdots + a_ng_n)) \\ &= U((a_0g_0 + \cdots + a_ng_n)(c_0), \ldots, (a_0g_0 + \cdots + a_ng_n)(c_n)) \\ &= U(a_0, \ldots, a_n) \\ &= a_0g_0 + \cdots + a_ng_n \\ &= f \end{aligned}$$

and

$$T(U(a_0, \ldots, a_n)) = T(a_0f_0 + \cdots + a_nf_n) = ((a_0f_0 + \cdots + a_nf_n)(c_0), \ldots, (a_0f_0 + \cdots + a_nf_n)(c_n)) = (a_0, \ldots, a_n)$$

where the coordinates of f in the basis $g_0, \ldots, g_n$ are $a_0, \ldots, a_n$. From this we can see that T is invertible with inverse U. Thus, since T is linear, it's an isomorphism.

3 Problems from §2.5

(a) Problem 1

— (a) —

False. The j^{th} column of Q is $[x_j]_\beta$.

— (b) —

True. It is the matrix representation of the identity transformation, which is invertible.

— (c) —

False. If Q changes β' coordinates into β coordinates, then

$$[T]_\beta = [\mathbb{I}T\mathbb{I}]_\beta = [\mathbb{I}]_{\beta\beta'}[T]_{\beta'}[\mathbb{I}]_{\beta'\beta} = Q^{-1}[T]_{\beta'}Q$$

— (d) —

False. The matrices $A, B \in \mathrm{M}_{n\times n}(\mathbb{F})$ are called similar if $B = Q^{-1}AQ$ for some $Q \in \mathrm{M}_{n\times n}(\mathbb{F})$.

— (e) —

True by Theorem 2.23 [1, p. 112]. In this case one can be retrieve from the other by the change of coordinate matrix.

(b) Problem 2

— (a) —

$\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

$$[\mathbb{I}]_{\beta\beta'} = \begin{pmatrix} [(a_1,a_2)]_\beta & [(b_1,b_2)]_\beta \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

— (b) —

$\beta = \{(-1,3), (2,-1)\}$ and $\beta' = \{(0,10), (5,0)\}$

$$[\mathbb{I}]_{\beta\beta'} = \begin{pmatrix} [(0,10)]_\beta & [(5,0)]_\beta \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

— (c) —

$\beta = \{(2,5), (-1,-3)\}$ and $\beta' = \{e_1, e_2\}$

$$[\mathbb{I}]_{\beta\beta'} = \begin{pmatrix} [(1,0)]_\beta & [(0,1)]_\beta \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}$$

— (d) —

$\beta = \{(-4,3),(2,-1)\}$ and $\beta' = \{(2,1),(-4,1)\}$

$$[\mathbb{I}]_{\beta\beta'} = \left(\begin{array}{cc} [(2,1)]_\beta & [(-4,1)]_\beta \end{array} \right) = \begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$$

(c) Problem 6 only a and b

For each of these, the matrix A is $[L_A]_\epsilon$ where ϵ is the standard basis.

— (a) —

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$Q = [\mathbb{I}]_{\epsilon\beta} = \left(\begin{array}{cc} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\epsilon & \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\epsilon \end{array} \right) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$Q^{-1} = [\mathbb{I}]_{\beta\epsilon} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$[L_A]_\beta = [\mathbb{I}]_{\beta\epsilon}[L_A]_\epsilon[\mathbb{I}]_{\epsilon\beta} = Q^{-1}AQ = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$$

— (b) —

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$$Q = [\mathbb{I}]_{\epsilon\beta} = \left(\begin{array}{cc} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\epsilon & \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]_\epsilon \end{array} \right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Q^{-1} = [\mathbb{I}]_{\beta\epsilon} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$[L_A]_\beta = [\mathbb{I}]_{\beta\epsilon}[L_A]_\epsilon[\mathbb{I}]_{\epsilon\beta} = Q^{-1}AQ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

4 §3.1 Problem 1

(a)

True. Performing an elementary row opertion does not change the dimensions of a matrix, and in this case we perform the operation on the $n \times n$ identity matrix.

(b)

This is generally false, since multiplying by any scalar of our choosing in the field over which the vector space lies is an elementary row operation, but if the field of scalars were $\mathbb{Z}_2$ consisting of only zero and one, then this property would hold.

(c)

True. Perform the "multiply any row by a scalar" row operation on the identity matrix with scalar of one.

(d)

False. The resulting matrix here is not elementary.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(e)

True. This is Theorem 3.2 [1, p. 150].

(f)

False. We can again use the counter-example from problem (d). The sums of those two matrices is the 2×2 matrix of all ones.

(g)

True.

(1) Exchanging row or column i with j of I_n will put ones at the ij and ji positions, and zeros at the ii and jj positions of the resulting matrix.

(2) Multiplying rows and columns of the identity matrix by a nonzero scalar results in a symmtric matrix.

(3) Multiplying row or column i by a scalar a and adding it to j is the "transpose operation" of multiplying row or column j by a and adding it to i.

(h)

False. Let A be

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

and B be the matrix resulting from swapping the two rows of A. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a+c & b+d \\ 0 & 0 \end{pmatrix}$$

we cannot find a matrix that will equal B via left multiplication by A, left alone an elementary matrix.

(i)

True. Since elementary row operations are invertible and their inverses are elementary row operations, if $B = EA$ then we have $E^{-1}B = A$ and E^{-1} is elementary.

5 Problems from §3.2

(a) Problem 2

We will use $\rightsquigarrow$ to denote the transformation of a matrix into an echelon form which gives us the necessary information about the rank.

— (a) —

The rank is two:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

— (b) —

The rank is three:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

— (c) —

The rank is two:

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

— (d) —

The rank is one since the second row is just a multiple of the first row.

— (e) —

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

— (f) —

The rank is three:

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

— (g) —

The rank is one since all rows are multiples of the first.

(b) Problem 3

Let A be an $m \times n$ matrix.

First assume that A has zero rank. This implies that L_A also has zero rank, that is, it has nullity of n since it's domain is $\mathbb{F}^n$. So $L_A(v) = 0$ for all $v \in \mathbb{F}^n$, in other words $Av' = 0$ for all $v' \in \mathrm{M}_{n\times 1}(\mathbb{F})$, but only the zero matrix does this. Hence A is the zero matrix.

Conversely, let A be the zero matrix. This has rank zero since L_A is the zero map, which has zero rank.

(c) Problem 4

— (a) —

Original

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 0 & -1 & 2 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$R_2 - 2R_1 \to R_2$
$R_3 - R_1 \to R_3$

$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_2 - C_1 \to C_2$
$C_3 - C_1 \to C_3$
$C_4 - 2C_1 \to C_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_3 - \frac{3}{2}C_2 \to C_3$
$C_4 - C_2 \to C_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\frac{-1}{2}R_2 \to R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the rank is two.

— (b) —

Original

$$\begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 1 \end{pmatrix}$$

$R_1 + 2R_2 \to R_1$
$R_3 + 2R_2 \to R_3$

$$\begin{pmatrix} 0 & 5 \\ -1 & 2 \\ 0 & 5 \end{pmatrix}$$

Swap R_1 and R_2

$$\begin{pmatrix} -1 & 2 \\ 0 & 5 \\ 0 & 5 \end{pmatrix}$$

$-1R_1 \to R_1$
$\frac{1}{5}R_2 \to R_2$
$\frac{1}{5}R_3 \to R_3$

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$R_1 + 2R_2 \to R_1$
$R_3 - R_2 \to R_3$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So the rank is two.

(d) Problem 7

We can row reduce A with the following steps

(1) $R_1 - R_2 \to R_1$

(2) $R_3 - R_2 \to R_3$

(3) Swap R_1,R_2

(4) $\frac{1}{2}R_2 \to R_2$

(5) $R_3 - R_2 \to R_3$

(6) $R_1 - R_3 \to R_1$

The elementary row matrices resulting from these operations, refer to E_i as the elementary row matrices for the i^{th} step above, can be inverted in order to obtain A as their product, since $E_6 \cdots E_1 A = I$. So we get $A = E_1^{-1} \cdots E_6^{-1}$. The inverses are (note $E_2 = E_5$):

$$E_1^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} E_2^{-1}, E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} E_3^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} E_6^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(e) Problem 20

— (a) —

The reduced row echelon form of A is the following.

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which informs us that the rank of A is three, i.e. the rank of L_A is three as well. This then tells us the nullity is 2 since the domain of the linear map is $\mathbb{F}^5$. Since L_A will map any item of its nullspace to zero the a matrix M with columns (coordinates) of vectors only from the nullspace of L_A will have that AM is zero. To boot, we know that we can find such an M with rank two because the dimension of the nullspace of L_A is two, i.e. the maximum size of any linearly independent set. So we'll choose the two vectors which correspond to the free variables from the reduced row echelon form of A above, namely

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So making M a matrix with all zero columns except for two columns which will be the columns immediately above will result in $AM = 0$.

Note that the ordering of the above columns in M is irrelevant since the matrix resulting from AM will have its i^{th} column be Am_i where m_i is the i^{th} column of M.

— (b) —

For any 5×5 matrix B with $AB = 0$, the columns of B must be contained within the nullspace of L_A. Since that space has dimension two, then no linear independent set of vectors can have a size greater than two. This includes the columns of B, so $\text{rank}(B)$ is two at most.

(f) Problem 21

Let A be an $m \times n$ matrix with rank m; denote by $0_{i\times j}$ the zero matrix in $\mathrm{M}_{i\times j}(\mathbb{F})$. So there are m columns of A which are linearly independent. This implies that through elementary column operations, i.e. multiplication on the right by elementary matrices, we can obtain the matrix $(I_m \; 0_{m\times(n-m)})$ where I_m is the $m \times m$ identity. Note that because A has m rows and is rank m, then $n \geq m$, so our previous subtraction is valid. Letting $E_1, \ldots, E_k$ be the matrices pertaining to the column operations on A, we have $AE_1 \cdots E_k = \left(I_m \; 0_{m\times(n-m)}\right)$, which yields

$$AE_1 \cdots E_k \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix} = \left(I_m \; 0_{m\times(n-m)}\right) \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix} = I_m$$

So

$$B = E_1 \cdots E_k \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix}$$

is the $n \times m$ matrix we require.

(g) Problem 22

Let B be an $n \times m$ matrix with rank m. By the same reasoning regarding columns in the previous problem, we perform elementary row operations on B to gain the following result.

$$E_k \cdots E_1 B = \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix}$$

Multiplying both sides by $\left(I_m \; 0_{m\times(n-m)}\right)$ leaves us with

$$\left(I_m \; 0_{m\times(n-m)}\right) E_k \cdots E_1 B = \left(I_m \; 0_{m\times(n-m)}\right) \begin{pmatrix} I_m \\ 0_{(n-m)\times m} \end{pmatrix} = I_m$$

and thus the $m \times n$ matrix A we are looking for is $\left(I_m \; 0_{m\times(n-m)}\right) E_k \cdots E_1$.

6

Let A be a $n \times n$ matrix such that $A^2 = 0$. Since the identity matrix, I, is also $n \times n$, then $I - A$ is also $n \times n$. Similarly $I + A$ is square of dimension n. Taking a look at the product of these two sums, we get

$$(I - A)(I + A) = (I - A)I + (I - A)A = I^2 - AI + IA - A^2 = I - A + A - 0 = I$$

as well as

$$(I + A)(I - A) = I(I - A) + A(I - A) = I^2 - IA + AI - A^2 = I - A + A - 0 = I.$$

So $I + A$ is the inverse of $I - A$.

7

(a)

Let $b, b' \in \mathbb{F}$ and $v, v' \in V$ with the following expressions in $\mathcal{B}$, $v = a_1\beta_1 + \cdots + a_n\beta_n$ and $v' = a'_1\beta_1 + \cdots + a'_n\beta_n$. For this we see that

$$\beta_i^*(bv + b'v') = \beta_i^*((ba_1 + b'a'_1)\beta_1 + \cdots + (ba_n + b'a'_n)\beta_n) = ba_i + b'a'_i = b\beta_i^*(v) + b'\beta_i^*(v')$$

and so β_i^* is linear.

(b)

The dual V^* has dimension n since it is the space $\mathcal{L}(V, \mathbb{F})$, so we need only show that either $\mathcal{B}^*$ is linearly independent or that it generates V^*. We choose generation.

Let $f \in V^*$. Since f is completely determined by its action on a basis of V, like $\mathcal{B}$, then we need express f in terms of $\mathcal{B}^*$ such that $f(\beta_i) = (a_1\beta_1^* + \cdots + a_n\beta_n^*)(\beta_i)$ where $a_1\beta_1^* + \cdots + a_n\beta_n^*$ is the yet-to-be-determined representaiton of f in $\mathcal{B}^*$. Since $\beta_i^*(\beta_j) = \delta_{ij}$ (the Kronecker-δ), then we simply need to set $a_i = f(\beta_i)$ to obtain our desired property.

(c)

Let $T : V \to W$ be a linear map. If we let $a, b \in \mathbb{F}$ and $U, V \in W^*$ then we have the following.

$$T^t(aU + bV) = (aU + bV) \circ T = aUT + bVT = aT^t(U) + bT^t(V)$$

So T^t is linear.

(d)

Let $\mathcal{B}$ be a basis for V, $\mathcal{C}$ be a basis for W, and $\mathcal{B}^*$, $\mathcal{C}^*$ be bases for V^*, W^*, respectively. Then for the linear map $T : V \to W$. The transpose of its coordinate matrix form $\mathcal{B}$ to $\mathcal{C}$ is $([T]_{\mathcal{CB}})^t$. Expanding we get

$$\begin{pmatrix} [T(\beta_1)]_{\mathcal{C}} \\ \vdots \\ [T(\beta_n)]_{\mathcal{C}} \end{pmatrix}$$

Given how we generate a linear functional with the dual basis $\mathcal{C}^*$ we get

$$\begin{pmatrix} c_1^*(T(\beta_1)) & \cdots & c_m^*(T(\beta_1)) \\ \vdots & \ddots & \vdots \\ c_1^*(T(\beta_n)) & \cdots & c_m^*(T(\beta_n)) \end{pmatrix}$$

which can be rewritten making use of T^t's definition

$$\begin{pmatrix} T^t(c_1^*)(\beta_1) & \cdots & T^t(c_m^*)(\beta_1) \\ \vdots & \ddots & \vdots \\ T^t(c_1^*)(\beta_n) & \cdots & T^t(c_m^*)(\beta_n) \end{pmatrix}$$

but for the way we defined the dual basis, $\mathcal{B}^*$, the ij entry of this matrix is i^{th} coordinate of $T^t(c_j^*)$ in $\mathcal{B}^*$. So the matrix is then

$$\begin{pmatrix} [T^t(c_1^*)]_{\mathcal{B}^*} & \cdots & [T^t(c_m^*)]_{\mathcal{B}^*} \end{pmatrix}$$

which is just $[T^t]_{\mathcal{B}^*\mathcal{C}^*}$

(e)

First assume that T is an isomorphism. Then the matrix $[T]_{\mathcal{C}\mathcal{B}}$ is invertible, which implies that its transpose is as well. However, its transpose is the matrix $[T^t]_{\mathcal{B}^*\mathcal{C}^*}$. so this too is invertible. Thus T^t is an isomorphism since it's linear.

Conversely, allow T^t to be an isomorphism. Then $[T^t]_{\mathcal{B}^*\mathcal{C}^*}$ is an invertible matrix. This is equal to the transpose of $[T]_{\mathcal{C}\mathcal{B}}$ and therfore T is an isomorphism since it is linear.

(f)

Define the map $M : V \to V^*$ by $M(a_1\beta_1 + \cdots + a_n\beta_n) = a_1\beta_1^* + \cdots + a_n\beta_n^*$. This definition demonstrates the map's linearity as well as it's surjective nature since $\mathcal{B}^*$ is a basis for $V*$. Now let $M(a_1\beta_1 + \cdots + a_n\beta_n) = M(a_1'\beta_1 + \cdots + a_n'\beta_n)$. Thus $a_1\beta_1^* + \cdots + a_n\beta_n^* = a_1'\beta_1^* + \cdots + a_n'\beta_n^*$, which can only be true if $a_i = a_i'$ for each i, since $\mathcal{B}^*$ is a basis for V^*. Hence M is injective. Given these three things, M is an isomorphism.

(g)

For $L : V \to V^{**}$ defined by $L(a_1\beta_1 + \cdots + a_n\beta_n) = a_1\beta_1^{**} + \cdots + a_n\beta_n^{**}$. This is simply the composition of two isomorphisms, namely $M_1 : V \to V^*$ and $M_2 : V^* \to V^{**}$ defined as in the previous problem. Thus L is an isomorphism.

(h)

Let $a, b \in \mathbb{F}$ and u, v be vectors in V. Letting G be any vector in V^* we get the following by its linearity (it's in $\mathcal{L}(V, \mathbb{F})$, remember) and by the definition of E_V itself.

$$\begin{aligned} E_V(au+bv)(G) &= G(au+bv) \\ &= aG(u) + bG(v) \\ &= a(E_V(u))(G) + b(E_V(v))(G) \\ &= (aE_V(u) + bE_V(v))(G) \end{aligned}$$

So E_V is linear.

(i)

Let $a_1\beta_1 + \cdots + a_n\beta_n$ be a vector in V. Then for any G in V^* we get the following by the newly found linearity of E_V.

$$\begin{aligned} E_V(a_1\beta_1 + \cdots + a_n\beta_n)(G) &= (a_1E_V(\beta_1) + \cdots + a_nE_V(\beta_n))(G) \\ &= (a_1E_V(\beta_1)(G) + \cdots + a_nE_V(\beta_n)(G)) \\ &= a_1G(\beta_1) + \cdots + a_nG(\beta_n) \end{aligned}$$

But for $G(\beta_i)$ is just the i^{th} coordinate of G in its $\mathcal{B}^*$ representation, i.e. $G(\beta_i)$ is $\beta_i^{**}(G)$. So the rightmost term from the above equation becomes

$$a_1\beta_i^{**}(G) + \cdots + a_n\beta_n^{**}(G)$$

and thus $E_V = L$.

(j)

Let $a_1\beta_1 + \cdots + a_n\beta_n$ be a vector in V. Then for any G in W^* we get the following through the use of the various definitions of the maps involved as well as their linearity.

$$\begin{aligned} ((T^{tt} \circ E_V)(a_1\beta_1 + \cdots + a_n\beta_n))(G) &= (E_V(a_1\beta_1 + \cdots + a_n\beta_n) \circ T^t)(G) \\ &= E_V(a_1\beta_1 + \cdots + a_n\beta_n)(G \circ T) \\ &= a_1(G \circ T)(\beta_1) + \cdots + a_n(G \circ T)(\beta_n) \\ &= E_W(a_1T(\beta_1) + \cdots + a_nT(\beta_n))(G) \\ &= ((E_W \circ T)(a_1\beta_1 + \cdots + a_n\beta_n))(G) \end{aligned}$$

So $T^{tt} \circ E_V = E_W \circ T$.

References

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