Math 500: Topology
Homework 1

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Problems

P-1

Let X be a topological space with some subset A such that for all $x \in A$ there exists an open set U such that

$$U \subset A \text{ and } x \in U \tag{P-1.1}$$

Let $\overline{U}$ be the set $\bigcup_{x \in A} U_x$ where U_x is a particular, yet arbitrary, set for which the property in P-1.1 holds with respect to x. As $\overline{U}$ is the union of subsets of A, clearly $\overline{U} \subset A$. Also for each $a \in A$, $a \in U_a \subset \overline{U}$, so $A \subset \overline{U}$. Thus A equals $\overline{U}$, a union of open sets, i.e. A is open.

P-2

Let X be some set with a collection of subsets, $\mathcal{T}$, such that $U \in \mathcal{T}$ if $X \setminus U$ is countable or is the set X itself. Since $X \setminus \emptyset$ is X, than $\emptyset$ in $\mathcal{T}$ and X is in $\mathcal{T}$ since its complement in itself is the empty set, which is countable.

Now let $\{U_\alpha\}$ be an arbitrary collection of sets in $\mathcal{T}$. The complement of the union of these sets, $X \setminus \bigcup_\alpha U_\alpha$, is equal to $\bigcap_\alpha (X \setminus U_\alpha)$ by DeMorgan, which has a cardinality that is no larger than that of its largest set. As each $X \setminus U_\alpha$ is countable, then $\bigcap_\alpha (X \setminus U_\alpha)$, and therefore $X \setminus \bigcup_\alpha U_\alpha$, is countable. So $\mathcal{T}$ is closed under arbitrary union of its elements.

Let $\{U_i\}$ be a finite collection of sets in $\mathcal{T}$. The complement of the intersection of these sets, $X \setminus \bigcap_i U_i A$, by DeMorgan, is also given by $\bigcup_i (X \setminus U_i)$. This set, however, is a countable union of countable sets and thus has countable cardinality. So $\mathcal{T}$ is closed under finite union of its elements.

With the above two properties of $\mathcal{T}$ and, as we saw, the fact that it contains $\emptyset$ and X, then $\mathcal{T}$ is a topology on X.

Exercises

E-1

Let X be a set with $\mathcal{A}$ a collection of subset of X. Allow $x \in X \setminus \bigcup_{A \in \mathcal{A}} A$. Then for all $A \in \mathcal{A}$, x is not in A. In other words x is in each and every $X \setminus A$; so x is in $\bigcap_{A \in \mathcal{A}} (X \setminus A)$. This gives us $X \setminus \bigcup_{A \in \mathcal{A}} A \subset \bigcap_{A \in \mathcal{A}} (X \setminus A)$.

Now let $x \in \bigcap_{A \in \mathcal{A}} (X \setminus A)$. Thus for all $A \in \mathcal{A}$, x is in $X \setminus A$. Then there is no A for which $x \in A$, so x is in $X \setminus \bigcup_{A \in \mathcal{A}} A$. Hence $\bigcap_{A \in \mathcal{A}} (X \setminus A) \subset X \setminus \bigcup_{A \in \mathcal{A}} A$.

With these two results, we have that

$$X \setminus \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (X \setminus A)$$

which we can use to prove that

$$X \setminus \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (X \setminus A).$$

Using a little complement-trickery, we have that $X \setminus \bigcap_{A \in \mathcal{A}} A$ is equal to $X \setminus \bigcap_{A \in \mathcal{A}} (X \setminus (X \setminus A))$, which from our above proof, we know to be equal to $X \setminus (X \setminus \bigcup_{A \in \mathcal{A}} (X \setminus A))$. However, this is simply $\bigcup_{A \in \mathcal{A}} (X \setminus A))$. Thus we have

$$X \setminus \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (X \setminus A)$$

as desired.

E-2

The following matrix has entries with values of $\subset$, $\supset$, $=$, and $\supset\subset$ standing for strictly finer than, strictly courser than, equal to, and not comparable. The entry at the i^{th}, j^{th} position corresponds to the relationship between the i^{th} and j^{th} subsets of the book's Figure 12.1. We will number the topologies in the book's figure from left-to-right, top-to-bottom. Given this, the relationships are as follows, leaving out the redundancies.

$$\begin{array}{ccccccccc}
= & \supset & \supset & \supset & \supset & \supset & \supset & \supset & \supset \\
 & = & \supset\subset & \supset\subset & \supset\subset & \supset\subset & \supset & \subset & \subset \\
 & & = & \supset & \supset\subset & \subset & \supset & \supset\subset & \subset \\
 & & & = & \supset\subset & \subset & \supset\subset & \subset & \subset \\
 & & & & = & \supset\subset & \supset\subset & \supset\subset & \subset \\
 & & & & & = & \supset & \supset\subset & \subset \\
 & & & & & & = & \subset & \subset \\
 & & & & & & & = & \subset \\
 & & & & & & & & =
\end{array}$$

E-3

(a)

Certainly $\emptyset$ and $\mathbb{R}$ are in $\mathcal{T}$ since the empty set satisfies the required property as it has no such x, and for $\mathbb{R}$ any interval that contains such an x will be contained in $\mathbb{R}$.

So let $\{U_\alpha\}$ be a subset of elements of $\mathcal{T}$ of arbitrary size. Allow x to be an element of $\bigcup_\alpha U_\alpha$. Then x is in some U_α and since $U_\alpha \in \mathcal{T}$, we can find an interval (a, b) such that $x \in (a, b) \subset U_\alpha$. Therefore (a, b) is also contained in $\bigcup_\alpha U_\alpha$ and so it's an element of $\mathcal{T}$. Thus $\mathcal{T}$ is closed under arbitrary union.

Now let $\{U_i\}$ be a finite collection of elements of $\mathcal{T}$. If $\bigcap_i U_i$ is the empty set, then its in $\mathcal{T}$ as per above, so let it be nonempty. Let $x \in \bigcap_i U_i$. Then there is a collection $\{(ab)_i\}$ of intervals such that for each i, $x \in (a, b)_i \subset U_i$. This then implies that $x \in \bigcap_i (a, b)_i$, but $\bigcap_i (a, b)_i \subset U_i$ for each i. Hence $x \in \bigcap_i (a, b)_i \subset \bigcap_i U_i$, and since $\bigcap_i (a, b)_i$ is an open interval, then $\bigcap_i U_i \in \mathcal{T}$; i.e. it is closed under finite union.

(b)

Let U be in the finite complement topology on $\mathbb{R}$. Then $X \setminus U$ is finite and denote it by $\{a_1, \ldots, a_n\}$ where $a_i < a_j$ for $i < j$. Let $x \in U$. Then either $x < a_1$, $x > a_n$, or $a_1 < x < a_n$. If $x < a_1$, then $x \in \left(x - 1, a_1 - \frac{|a_1 - x|}{2}\right) \subset U$ and likewise $x \in \left(a_n + \frac{|a_n - x|}{2}, x + 1\right) \subset U$ when $x > a_n$. Now let $a_1 < x < a_n$. Here we can find i such that a_i is the greatest element in $X \setminus U$ which is less than x. With this element we have that $x \in \left(a_i + \frac{|a_i - x|}{2}, a_{i+1} - \frac{|a_{i+1} - x|}{2}\right) \subset U$. Thus in all three cases we can find an interval contained in U, so U must also be in $\mathcal{T}$. Hence $\mathcal{T}$ is finer than the finite complement topology.