Math 500: Topology
Homework 3

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Problems

P-1

In each of the following subproblems, let X and Y be the orginal topological spaces on which f is defined and $\overline{X}$ or $\overline{Y}$ be the respective spaces which are alterted as per the subproblem description.

(a) Make X finer.

Making the domain finer won't affect the continuity. Let U be an open set of Y. Then $f^{-1}(U)$ is open in X. But since $\overline{X}$ is finer than X, then $\mathcal{T}_X \subset \mathcal{T}_{\overline{X}}$ and therefore $f^{-1}(U)$ is also open in $\overline{X}$.

(b) Make X courser.

Making the domain courser can, but will not necessarily, result in f not being continuous. As an example of a function "losing its continuity": if X and Y are both the discrete topologies on $\mathbb{R}$, and f is the identity map, then making X courser by changing it to the indiscrete topology will make $f^{-1}(U)$ non-open if U is any proper, nontrivial subset of Y. On the other hand, f can retain its continuity, as exampled by the following scenario. Let X be the discrete topology on $\mathbb{R}$ and Y be the indiscrete, with f again being the identity map. In this case, no matter how course X is made, f will always be continuous.

(c) Make Y finer.

Again making the topology of Y finer can, but will not necessarily, cause f to lose its continuity. An example when it does is if f is the identity map an X and Y are the same topological spaces, then adding any set to the topology on Y (and any other sets necessary to maintain it topological status) will cause f to no longer be continuous. An example of where f does not lose continuity is if X and Y are the same sets, X has the discrete topology, Y has any other except for the discrete, and f is the idenity map. Then in this case X has the "finest" topology for the set $X = Y$, so no matter what sets are added to the topology on Y to make it finer, no set added will be added that isn't already in the discrete topology.

(d) Make X courser.

Making Y courser will not affect the continuity of f. This is so since $\mathcal{T}_{\overline{Y}} \subset \mathcal{T}_Y$ and every $U\mathcal{T}_Y$ has that $f^{-1}(U)$ is open in X, so any set of $\mathcal{T}_{\overline{Y}}$ will have the same.

P-2

Using the objects in the images of Figure 1 we have the following homeomorphism classes.

saucer $\equiv$ glass $\equiv$ spoon $\equiv$ fork $\equiv$ plate $\equiv$ coin $\equiv$ nail $\equiv$ bolt

cup $\equiv$ nut $\equiv$ wedding ring $\equiv$ flower pot $\equiv$ key

(a) cup [1]

(b) saucer [?]

(c) glass [3]

(d) spoon [4]

(e) fork [5]

(f) plate [6]

(g) coin [7]

(h) nail [8]

(i) bolt [9]

(j) nut [10]

(k) wedding ring [11]

(l) flower pot [12]

(m) key [13]

Figure 1: Images of items to partition into homeomorphic equivalency classes.

P-3

Here we can use polar coordinates to convert between the disk and the square. Basically, a point (r, θ) in the square will be the point in the disk of radius r away from the origin, and at an angle θ from the positive x-axis. Given this we define our map $f : D^2 \to I^2$ as follows[1]

$$f(x, y) = (\sqrt{x^2 + y^2}, \overline{\arctan}(y, x))$$

begetting an inverse function of

$$f^{-1}(r, \theta) = (r \cos\theta, r \sin\theta)$$

Since each of the composite functions which make up f are individually continuous for $x + y \leq 1$ then the indivial components of f are each continuous by Munkres Theorem 18.2 (c) which in turn gives us, by Munkres Theorem 18.4, that f itself is continuous. An identical argument holds for f^{-1}. Because f^{-1} is continuous, then f is open. So because f is an invertible, open, continuous map, than it is a homeomorphism, and thus D^2 and I^2 are homeomorphic.

P-4 Munkres §18 exercise 1

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous according to the open set definition. Let $x \in \mathbb{R}$ and $\epsilon > 0$, then $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open. This means that there exists some interval contained inside it which contains x, i.e. there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$. Thus we have that any y in $(x - \delta, x + \delta)$ will also be in $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$, and hence $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. Thus f is continuous according to the $\epsilon - \delta$ definition.

Conversely assume that the $\epsilon - \delta$ definition of continuity holds for f. Let V be open in $\mathbb{R}$, then for each $x \in f^{-1}(V)$ there is an ϵ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset V$. From the $\epsilon - \delta$ property of f we get that $f((x - \delta, x + \delta)) \subset (f(x) - \epsilon, f(x) + \epsilon) \subset V$, which implies that $(x - \delta, x + \delta) \subset f^{-1}(V)$, i.e. $f^{-1}(V)$ is open. Thus f is continuous according to the set definition.

P-5 Prove Munkres' §18 Theorem 1

To prove the equivalency of this theorem, we will proceed by proving

(a) (1) $\implies$ (3)

(b) (3) $\implies$ (2)

(c) (2) $\implies$ (1)

(d) (1) $\implies$ (4)

(e) (4) $\implies$ (1)

and in each case $f : X \to Y$ will be a function with X and Y topological spaces.

(a) (1) $\implies$ (3)

Assume that f is a continuous function. Let $B \in Y$ be closed. Then $Y \setminus B$ is open. Therefore $f^{-1}(Y \setminus B)$ is as well, but this is equal to $X \setminus f^{-1}(B)$, and so $f^{-1}(B)$ must be closed.

[1]The function we name $\overline{\arctan}$ is just the inverse tangent function which takes the different quandrants into account. We will assume it returns the value of the angle from the positive x-axis in the range $[0, 2\pi)$. The details of the function are out of scope of the proof, but we note that such functions exist as this C function: `http://www.cplusplus.com/reference/clibrary/cmath/atan2/`

(b) (3) $\implies$ (2)

Let $f : X \to Y$ be a function such that for all closed sets B in Y, $f^{-1}(B)$ is closed. For $A \subset X$, $A \subset f^{-1}(f(A))$ is always true and since sets are subsets of their own closure, then $A \subset f^{-1}(\overline{f(A)})$. Since $\overline{f(A)}$ is a closed set, then by assumption $f^{-1}(\overline{f(A)})$ is closed, but because it contains A, then it contains $\overline{A}$ since the closure of A is the union of closed supersets of A. So we have $\overline{A} \subset f^{-1}(\overline{f(A)})$, implying $f(\overline{A}) \subset \overline{f(A)}$.

(c) (2) $\implies$ (1)

Assume that for all $A \subset X$, $f(\overline{A}) = \overline{f(A)}$. Let U be an open set of Y. Let $x \in \overline{X \setminus f^{-1}(U)}$, which implies that $f(x) \in f\left(\overline{X \setminus f^{-1}(U)}\right)$. Since $Y \setminus U$ is closed we have

$$f\left(\overline{X \setminus f^{-1}(U)}\right) \subset \overline{f(X \setminus f^{-1}(U))} = \overline{f(f^{-1}(Y) \setminus f^{-1}(U))} = \overline{f(f^{-1}(Y \setminus U))} \subset \overline{Y \setminus U} = Y \setminus U$$

and from it we get $x \in f^{-1}(Y \setminus U)$, but $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ and so $x \in X \setminus f^{-1}(U)$. Thus $\overline{X \setminus f^{-1}(U)} \subset X \setminus f^{-1}(U)$, and therefore, since a set is a subset of its own closure, $\overline{X \setminus f^{-1}(U)} = X \setminus f^{-1}(U)$, so $X \setminus f^{-1}(U)$ is closed. By this $f^{-1}(U)$ is open, which yields that f is continuous.

(d) (1) $\implies$ (4)

Assume that $f : X \to Y$ is a continuous function. Let $V \subset Y$ be a neighborhood of $f(x)$ for some $x \in X$. Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is open. Since $f(f^{-1}(V)) \subset V$ is always true, then $f^{-1}(V)$ is a neighborhood U of x with $f(U) \subset V$.

(e) (4) $\implies$ (1)

Assume that for all neighborhoods V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. Let V be an open set of Y. For each $x \in f^{-1}(V)$ let U_x denote a neighborhood of x such that $f(U_x) \subset V$, i.e. $U_x \subset f^{-1}(V)$. Therefore $\cup_{x \in f^{-1}(V)} U_x \subset f^{-1}(V)$, but since each U_x contains x then $f^{-1}(V) \subset \cup_{x \in f^{-1}(V)} U_x \subset f^{-1}(V)$. Therefore $f^{-1}(V)$ equals $\cup_{x \in f^{-1}(V)} U_x$ which, as a union of open sets, is open. Hence f is continuous.

P-6 Munkres §19 exercise 7

By $\mathbb{R}_n$ denote the set

$$\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times (\mathbb{R} \setminus \{0\})}_{n \text{ terms}} \times \{0\} \times \{0\} \cdots$$

Note that with this notation $\mathbb{R}_0$ is the product containing only singletons of zero. So then, we can represent $\mathbb{R}^\infty$ by

$$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}_0} \mathbb{R}_n$$

So in light of Munkres Theorem 19.5,

$$\overline{\mathbb{R}_n} = \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \cdots \times \left(\overline{\mathbb{R} \setminus \{0\}}\right) \times \overline{\{0\}} \times \overline{\{0\}} \cdots = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \times \{0\} \times \{0\} \cdots$$

for both the box and product topologies. This gives us that

$$\overline{\mathbb{R}^\infty} = \bigcup_{n \in \mathbb{N}_0} \overline{\mathbb{R}_n}$$

which simply implies that the closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\omega$ for both the box and product topologies.

References

[1] Cup image Figure 1a http://img1.123freevectors.com/wp-content/uploads/objects_big/067_objects_coffee-cup-free-vector.jpg

[2] saucer image figure 1b: http://www.bryanchina.com/Mugs/BWE-066%20Cappuccino.Espresso%20Cappuccino%20Saucer%20White.JPG

[3] glass image figure 1c: http://party.rainbow-rental.com/dinnerware/dinnerware_images/highball.jpg

[4] spoon image figure 1d: http://iblogwhatihear.com/wp-content/uploads/2010/01/spoon.jpg

[5] fork image figure 1e: http://www.ccesonline.com/images/fork260.jpg

[6] plate image figure 1f: http://9pin.in/images/designer-photo-plate-room-tea.jpg

[7] coin image figure 1g: http://www.marshu.com/articles/images-website/articles/presidents-on-coins/quarter-coin-head.jpg

[8] nail image figure 1h: http://image.tradevv.com/2010/03/26/zjlongtong1_1080859_600/stainless-steel-nail.jpg

[9] bolt image figure 1i: http://us.123rf.com/400wm/400/400/moori/moori0803/moori080300178/2744907-used-metal-bolt-on-a-white-background.jpg

[10] nut image figure 1j: http://www.portlandbolt.com/image/products/full/heavy_hex_nut1.jpg

[11] wedding ring image figure 1k: http://images.pictureshunt.com/pics/w/wedding_ring-2165.jpg

[12] flower pot image figure 1l: http://cchs.usd224.com/Classes09/Flowersforless/FlowerPot.jpg

[13] key image figure 1m: http://www.feelnumb.com/wp-content/uploads/2009/03/keyHorizontal.jpg