Math 500: Topology
Homework 5

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Problems

1 Munkres §26 exercise 8

Let $f : X \to Y$ be a function with Y a compact Hausdorff space, and define the graph of f to be the set

$$G_f = \{x \times f(x) \mid x \in X\}$$

Continuity of f implies G_f is closed. Assume that f is a continuous map. All elements of the set $X \times Y \setminus G_f$ have the form $(x, f(x'))$ where $x \neq x'$ given the definition of G_f. Let $N = U \times V$ be a neighborhood of $(x, f(x'))$. If there is an x_0 such that $(x_0, f(x_0))$ is contained within N then we can separate $f(x_0)$ and $f(x')$ with open sets according to the Hausdorff condition of Y, and replace V with V' where V' is the neighborhood containing $f(x')$. This yields a new neighorhood $N' = U \times V'$ of $(x, f(x'))$ which doesn't contain $(x_0, f(x_0))$. Doing this for all such $(x_0, f(x_0))$ in N will return to us a neighborhood of $(x, f(x'))$ which is disjoint from G_f. Thus $X \times Y \setminus G_f$ is open, and therefore G_f is closed.[1]

G_f being closed implies continuity of f. Assume that G_f is a closed subset of $X \times Y$. Let V be a neighborhood of $f(x_0)$ for some point $x_0 \in X$. Then $Y \setminus V$ is a closed set yielding that $B = G_f \cap (X \times (Y \setminus V))$ is also closed. Since Y is compact, then by Munkres exercise 26.7 $\pi_1(B)$ is closed as well. Now because B is the set of all points $x \times f(x)$ where $f(x) \notin V$, then $U = X \setminus \pi_1(B)$ is a neighborhood of x_0 as $f(x_0) \in V$ and furthermore $U \in f(X)$. Thus by Munkres Theorem 18.3 gives us that f is continuous.

2 Munkres §27 exercise 5

Some thoughts for this proof where inspired by Rudin's proof of the uncountability of nonempty perfect sets in $\mathbb{R}^n$ [1, Theorem 2.43 pg. 41]

Let $\{A_n\}$ be a countable collection of closed subsets of a compact Hausdorff set X. Assume that each set in the collection has empty interior. For later contradiction assume that $\cup A_n$ has nonempty interior, U. Let $\{x_n\}$ be a sequence of points where each x_n is some point of A_n. We associate with this sequence, a sequence of subsets of U where V_1 is an open, nonempty subset of U such that $x_1 \notin \overline{V_1}$ and each V_n is an open, nonempty subset of V_{n-1} such that $x_n \notin \overline{V_n}$. We can find such a V_n for each x_n because of the foundation layed for us by step one of Munkres proof of Theorem 27.7 for a compact Hausdorff space such as X. Note that we need not concern ourselves with whether or not each x_n is an isolated point as it is not contained in U since each A_n has no interior.

Now since each element of $\{V_n\}$ is nonempty and $V_n \subset V_{n-1} \subset \cdots \subset V_1$, then each element of $\{\overline{V_n}\}$ is also nonempty and $\overline{V_n} \subset \overline{V_{n-1}} \subset \cdots \subset \overline{V_1}$ yielding that $\{\overline{V_n}\}$ is a collection of closed sets with the finite intersection property. Since our space, X, is compact $V = \cap \overline{V_n}$ is therefore nonempty.

However, since each x_n is not in V, then by Munkres Lemma 26.4 we can find an open U_n containing V with $x_n \notin U_n$. This implies that no limit point of V can be contained in any A_n, but since each A_n has no interior than no interior point of V can be contained within any A_n. Thus V has no interesection with $\cup A_n$, which contradicts the nonemptiness of V since $V \subset \cup A_n$. Therefore we must have that the interior of $\cup A_n$, U, is empty.

3 Munkres §28 exercise 6

Let f be an isometry on a compact metric space X.

f is continuous. An isometry is continuous, easily seen by the plain, old $\epsilon\delta-$definition of continuity on a metric space since for any ϵ we can set $\delta = \epsilon$ and we will get that

$$d(x, y) < \delta \implies (f(x), f(y)) < \epsilon$$

since $d(x, y) = d(f(x), f(y))$.

[1] I didn't use the continuity of f, which scares me, but it doesn't look like anything is wrong. Please advise.

f is injective If x and y are in X such that $f(x) = f(y)$, then we have that $d(f(x), f(y) = 0$, but because f is an isometry, then this yields $d(x, y) = 0$ which can only be the case if $x = y$.

f is surjective For later contradiction, assume that f is not surjective. Then there is some $a \in X$ with $a \notin f(X)$. Then since X is Hausdorff and $f(X)$ is compact (it's the continuous image of X), then Munkres Lemma 26.4 allows us to find an ϵ such that the ϵ-neighborhood of a is disjoint from $f(X)$. With this in mind, we construct a sequence recursively by defining x_n to be $f(x_{n-1})$ and with a base case of $x_1 = a$. Hence for any points x_i, x_j of the sequence with $i < j$

$$d(x_i, x_j) = d(f(x_{i-1}), f(x_{j-1})) = d(f \circ f(x_{i-2}), f \circ f(x_{j-2})) \cdots = d(\underbrace{f \circ \cdots \circ f}_{i-1}(a), \underbrace{f \circ \cdots \circ f}_{j-1}(a))$$

by the definition of the sequence, and has

$$d(\underbrace{f \circ \cdots \circ f}_{i-1}(a), \underbrace{f \circ \cdots \circ f}_{j-1}(a)) = d(a, \underbrace{f \circ \cdots \circ f}_{j-i}(a))$$

by the isometric property of f. However, the distance from a to any point of the image of f is greater than or equal to ϵ, so $d(x_i, x_j) \geq \epsilon$, indicating that this particular sequence has no limit point. This contradicts the compactnes of X as it would imply that X would not be limit-point compact. Hence f must be surjective.

Finally, we have that f is a continuous bijection from X to itself. This gives us that f^-1 too is continuous, and so f is a homeomorphism.

References

[1] Rudin, Walter. *Principles of Mathematical Analysis* 3rd Ed. McGraw-Hill: New York, 1964.