Math 500: Topology
Homework 7

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January 12, 2013
http://coursework.tylerlogic.com/courses/math500/homework07

(1) $\mathbb{R}^n$

  (a) **connected:** Yes, it is the finite product of a connected set, $\mathbb{R}$.
  (b) **path connected:** Yes, the straight line between any two points will be continuous.
  (c) **metrizable:** Yes, using the euclidean or square metric. Munkres Theorem 20.3
  (d) **compact:** No, $\mathbb{R}$ is not even compact, so $\mathbb{R}^n$ is not.
  (e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.
  (f) $\boldsymbol{T_2}$**:** Yes, it's metrizable.
  (g) $\boldsymbol{T_3}$**:** Yes, it's metrizable.
  (h) $\boldsymbol{T_4}$**:** Yes, it's metrizable.
  (i) **first-countable:** Yes, it's metrizable.
  (j) **second-countable:** Yes, it is the finite product of the second countable set $\mathbb{R}$.
  (k) **locally Euclidean:** Yes, it is $\mathbb{R}^n$.

(2) $\mathbb{R}_d$

  (a) **connected:** No, every set is clopen.
  (b) **path connected:** No, because its not connected.
  (c) **metrizable:** Yes, given by the discrete metric.
  (d) **compact:** No, this is finer than $\mathbb{R}$ and $\mathbb{R}$ is not compact.
  (e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.
  (f) $\boldsymbol{T_2}$**:** Yes, it's metrizable.
  (g) $\boldsymbol{T_3}$**:** Yes, it's metrizable.
  (h) $\boldsymbol{T_4}$**:** Yes, it's metrizable.
  (i) **first-countable:** Yes, it's metrizable.
  (j) **second-countable:** No, it is finer than $\mathbb{R}_\ell$ and that is not second-countable.
  (k) **locally Euclidean:** Yes, any open set of $\mathbb{R}$ will be an open set here. So every open neighborhood of any element of $\mathbb{R}_d$ has a neighborhood which is open in $\mathbb{R}$.

(3) $\mathbb{R}_\ell$

  (a) **connected:** No, $[-\infty, x)$ and $[x, +\infty)$ are open sets that separate the space for any $x \in \mathbb{R}_\ell$.
  (b) **path connected:** No, because it is not connected.
  (c) **metrizable:** Yes, in each lower limit basis element we can fit an open ball of the euclidean metric, and vice versa.
  (d) **compact:** No, $\{[-n, n) \mid n \in \mathbb{Z}^+\}$ has no finite subcover.
  (e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.
  (f) $\boldsymbol{T_2}$**:** Yes, it's metrizable.
  (g) $\boldsymbol{T_3}$**:** Yes, it's metrizable.
  (h) $\boldsymbol{T_4}$**:** Yes, it's metrizable.
  (i) **first-countable:** Yes, it's metrizable.
  (j) **second-countable:** No, section 30 example 3.
  (k) **locally Euclidean:** Yes, locally 1-euclidean. For any $x \in \mathbb{R}_\ell$ the set $\cup_n[x - 1 + \frac{1}{n}, x + 1)$ contains x, is open in $\mathbb{R}_\ell$, and equals the open set of $\mathbb{R}$, $(x - 1, x + 1)$.

(4) $\mathbb{R}_{fc}$

  (a) **connected:** Yes, a non-trivial clopen set would need to be finite and infinite, which isn't possible. Thus there are no non-trivial clopen sets.
  (b) **path connected:** Yes, the constant function on $\mathbb{R}$ will satisfy the requirements.

(c) **metrizable:** No, not Hausdorff.

(d) **compact:** Yes, any element of an open cover, $\mathcal{A}$, will have a finite number of elements not contained in it, so we only need to select a finite number of other elements from $\mathcal{A}$ to cover the entire space.

(e) $\boldsymbol{T_1}$**:** Yes, one point sets are finite, so their complements are open, so they are closed.

(f) $\boldsymbol{T_2}$**:** No, since no open sets in $\mathbb{R}_fc$ are disjoint.

(g) $\boldsymbol{T_3}$**:** No, not Hausdorff.

(h) $\boldsymbol{T_4}$**:** No, not Hausdorff.

(i) **first-countable:** No. Assume for contradiction that $\{A_i\}$ is a countable basis at x. Then $U = \cup_i \{\mathbb{R} \setminus A_i\}$ is a countable collection of points since it is a countable union of finite sets. Then for any $y \in \mathbb{R} \setminus U$ ($\mathbb{R} \setminus U$ is nonempty since $\mathbb{R}$ is uncountable), $\mathbb{R} \setminus \{y\}$ is open in $\mathbb{R}_{fc}$ but contains no A_i.

(j) **second-countable:** No, since it is not first countable.

(k) **locally Euclidean:** Yes, every neighborhood of a point x is an open set in $\mathbb{R}$ since finite sets are closed in $\mathbb{R}$. Thus each x has a neighborhood homeomorphic to an open set of $\mathbb{R}$.

(5) S^1

(a) **connected:** Yes, it is path connected.

(b) **path connected:** Yes, Section 24 example 5.

(c) **metrizable:** Yes, the magnitude of the acute angle between two points in the set.

(d) **compact:** Yes, closed and bounded subset of $\mathbb{R}^2$, so compact by Heine-Borel.

(e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.

(f) $\boldsymbol{T_2}$**:** Yes, it's a subspace of the Hausdorff space $\mathbb{R}^2$.

(g) $\boldsymbol{T_3}$**:** Yes, it's normal.

(h) $\boldsymbol{T_4}$**:** Yes, it's Hausdorff and compact.

(i) **first-countable:** Yes, it's metrizable.

(j) **second-countable:** Yes, it's a subspace of a second countable space, $\mathbb{R}^2$.

(k) **locally Euclidean:** Yes, it's a subspace of a $\mathbb{R}^2$.

(6) $\mathbb{R} \times \mathbb{R}$, dictionary

(a) **connected:** Yes, it is path connected.

(b) **path connected:** Yes, the straight line between any two points will be continuous.

(c) **metrizable:** Yes, Section 50 supplementary exercise 4.

(d) **compact:** No, since it is Hausdorff, being compact would imply that it is an m-manifold, but it is not due to Section 50 exercise 2.

(e) $\boldsymbol{T_1}$**:** Yes, this is locally euclidean.

(f) $\boldsymbol{T_2}$**:** Yes, this is normal.

(g) $\boldsymbol{T_3}$**:** Yes, this is normal.

(h) $\boldsymbol{T_4}$**:** Yes, this is an order topology.

(i) **first-countable:** Yes, it's metrizable.

(j) **second-countable:** No, Section 50 exercise 4.

(k) **locally Euclidean:** Yes, it is 1-euclidean.

(7) $\mathbb{R}^\omega$, product topology

(a) **connected:** Yes, Section 23 example 7.

(b) **path connected:** ???

(c) **metrizable:** Yes, Munkres Theorem 20.5.

(d) **compact:** No, $\mathbb{R}$ is not compact.

(e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.
(f) $\boldsymbol{T_2}$**:** Yes, it's metrizable.
(g) $\boldsymbol{T_3}$**:** Yes, it's metrizable.
(h) $\boldsymbol{T_4}$**:** Yes, it's metrizable.
(i) **first-countable:** Yes, it's metrizable.
(j) **second-countable:** Yes, since $\mathbb{R}$ is second countable, by Munkres Theorem 30.2.
(k) **locally Euclidean:** Yes, since it's metrizable.

(8) *TSC*

(a) **connected:** Yes, Section 24 example 7.
(b) **path connected:** No, Section 24 example 7.
(c) **metrizable:** Yes, it's regular and second countable. (Urysohn's metrization theorem).
(d) **compact:** Yes, it is a closed and bounded subset of $\mathbb{R}^2$.
(e) $\boldsymbol{T_1}$**:** Yes, it's metrizable.
(f) $\boldsymbol{T_2}$**:** Yes, it's a subspace of the Hausdorff space $\mathbb{R}^2$.
(g) $\boldsymbol{T_3}$**:** Yes, it's normal.
(h) $\boldsymbol{T_4}$**:** Yes, it's compact and Hausdorff.
(i) **first-countable:** Yes, it's metrizable.
(j) **second-countable:** Yes, it's a subspace of the second countable space $\mathbb{R}^2$.
(k) **locally Euclidean:** Yes, it's a subspace of $\mathbb{R}^2$.