Math 501: Differential Geometry
Homework 1

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January 26, 2013
http://coursework.tylerlogic.com/courses/math501/homework01

1 do Carmo pg. 5 exercise 2

Assume for later contradiction that $\alpha'(t_0)$ is not orthogonal to $\alpha(t_0)$. Since $\alpha'(t_0)$ is nonzero, then we are able to find a $\epsilon > 0$ such that either $|\alpha(t_0 - \epsilon)| < |\alpha(t_0)|$ or $|\alpha(t_0 + \epsilon)| < |\alpha(t_0)|$. However, either case contradicts the fact that $\alpha(t_0)$ is the point of the trace of α which is closest to the origin. Thus $\alpha'(t_0) \perp \alpha(t_0)$ must be true.

2 do Carmo pg. 7 exercise 2

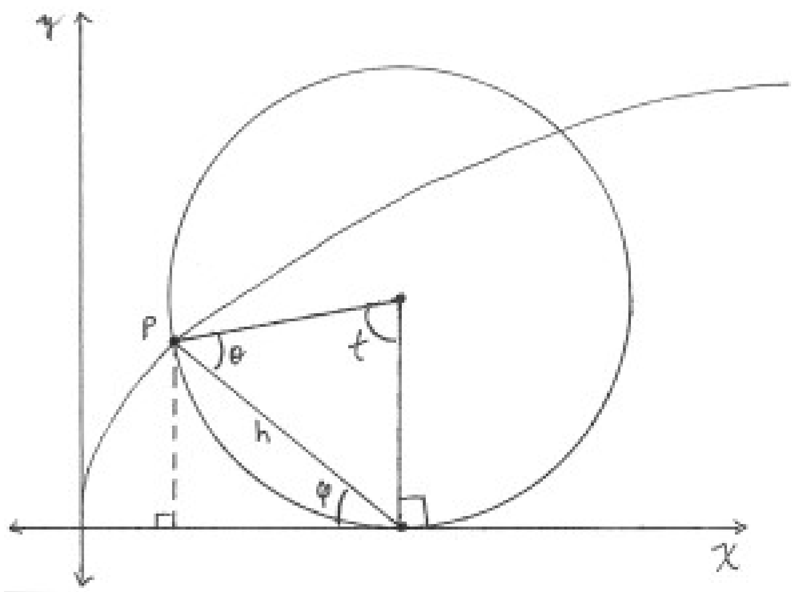

Figure 1: Cycloid at a Glance.

(a)

We will parametrize the cycloid by the angle of rotation, denote it by t. From the diagram in Figure 1 it can be seen that

$$\begin{aligned} \theta &= \frac{1}{2}(\pi - t) \\ \varphi &= \frac{\pi}{2} - \theta = \frac{t}{2} \end{aligned}$$

Thus the value of h is determined by the following sequence of equations.

$$\begin{aligned} \frac{h}{2} &= \cos\theta \\ h &= 2\cos\theta \\ h &= 2\cos\left(\frac{1}{2}(\pi - t)\right) \\ h &= 2\sin\frac{t}{2} \end{aligned}$$

With these things, we can finally derive the point p.

$$\begin{aligned} p_x &= t - h\cos\varphi \\ p_x &= t - 2\sin\frac{t}{2}\cos\frac{t}{2} \\ p_x &= t - \sin\ t \end{aligned}$$

and

$$\begin{aligned} p_y &= h\sin\varphi \\ p_y &= \left(2\sin\frac{t}{2}\right)\sin\frac{t}{2} \\ p_y &= 2\sin^2\frac{t}{2} \\ p_y &= 1 - \cos\ t \end{aligned}$$

(b)

Let's rotate from 0 to 2π, fixing t_0 at 0. By making use of the double angle formula for cos, we have the following.

$$\begin{aligned} s(2\pi) &= \int_0^{2\pi} |\alpha'(t)|\ dt \\ &= \int_0^{2\pi} |(1-\cos t, \sin t|\ dt \\ &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\ dt \\ &= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t}\ dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\ dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2}\right)}\ dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos^2\frac{t}{2} + \sin^2\frac{t}{2}}\ dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{t}{2}}\ dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\ dt \\ &= -4\cos\frac{t}{2}\Big|_0^{2\pi} \\ &= -4\cos\pi + 4\cos(0) \\ &= 8 \end{aligned}$$

3 do Carmo pg. 7 exercise 3

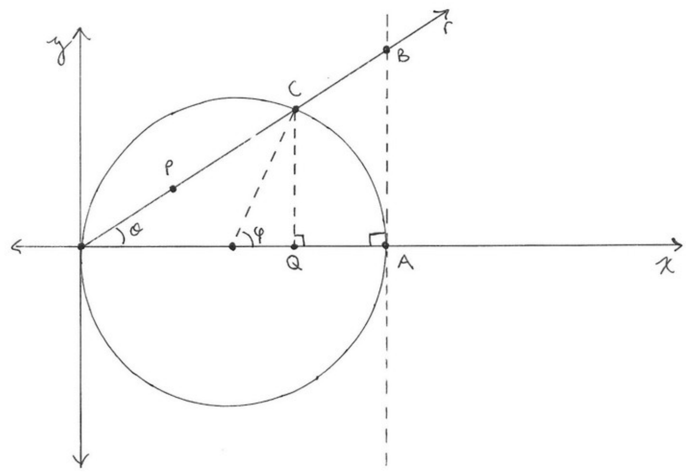

Figure 2: Cissoid of Diocles at a Glance.

(a)

For this problem we will use Figure 2 as a guide. Using some trigonometry, from the figure we get the following (remember the diameter is length $2a$)

$$\begin{aligned} \varphi &= 2\theta \\ OB &= \frac{OA}{\cos\theta} = \frac{2a}{\cos\theta} \\ OC &= \frac{a\sin 2\varphi}{\sin\theta} = \frac{a\sin 2\theta}{\sin\theta} \end{aligned}$$

These allow us to have the following

$$\begin{aligned} p &= (OP\cos\theta, OP\sin\theta) \\ &= (CB\cos\theta, CB\sin\theta) \\ &= (OB - OC)\cos\theta, (OB - OC)\sin\theta) \\ &= \left(\left(\frac{2a}{\cos\theta} - \frac{a\sin 2\theta}{\sin\theta}\right)\cos\theta, \left(\frac{2a}{\cos\theta} - \frac{a\sin 2\theta}{\sin\theta}\right)\sin\theta\right) \\ &= \left(\left(\frac{2a}{\cos\theta} - \frac{2a\cos\theta\sin\theta}{\sin\theta}\right)\cos\theta, \left(\frac{2a}{\cos\theta} - \frac{2a\cos\theta\sin\theta}{\sin\theta}\right)\sin\theta\right) \\ &= \left(2a - 2a\cos^2\theta, \frac{2a\sin\theta}{\cos\theta} - 2a\cos\theta\sin\theta\right) \\ &= \left(2a\sin^2\theta, \frac{2a\sin\theta - 2a\cos^2\theta\sin\theta}{\cos\theta}\right) \\ &= \left(\frac{2a\sin^2\theta\sec^2\theta}{\sec^2\theta}, \frac{2a\sin\theta(1-\cos^2\theta)}{\cos\theta}\right) \\ &= \left(\frac{2a\sin^2\theta}{\sec^2\theta\cos^2\theta}, \frac{2a\tan\theta\sin^2\theta\sec^2\theta}{\sec^2\theta}\right) \\ &= \left(\frac{2a\tan^2\theta}{1+\tan^2\theta}, \frac{2a\tan\theta\sin^2\theta}{(1+\tan^2\theta)\cos^2\theta}\right) \\ &= \left(\frac{2a\tan^2\theta}{1+\tan^2\theta}, \frac{2a\tan^3\theta}{1+\tan^2\theta}\right) \end{aligned}$$

Hence letting $t = \tan\theta$ we get

$$p = \left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right)$$

(b)

Because

$$\alpha'(t) = \left(\frac{4at}{1+t^2} - \frac{4at^3}{(1+t^2)^2}, \frac{6at^2}{1+t^2} - \frac{4at^4}{(1+t^2)^2}\right) = \left(\frac{4at}{(1+t^2)^2}, \frac{2at^2(3+t^2)}{(1+t^2)^2}\right)$$

then $\alpha(0) = (0,0)$, and so α has a singular point at the origin $(0,0)$.

(c)

$$\begin{aligned}\lim_{t\to\infty} \alpha(t) &= \lim_{t\to\infty}\left(\frac{2at^2}{1+t^2}, \frac{2at^3}{1+t^2}\right) \\ &= \left(2a \lim_{t\to\infty} \frac{t^2}{1+t^2}, 2a \lim_{t\to\infty} \frac{t^3}{1+t^2}\right) \\ &= (2a, \infty)\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} \alpha'(t) &= \lim_{t\to\infty}\left(\frac{4at}{(1+t^2)^2}, \frac{2at^2(3+t^2)}{(1+t^2)^2}\right) \\ &= \left(4a \lim_{t\to\infty} \frac{t}{(1+t^2)^2}, 2a \lim_{t\to\infty} \frac{t^2(3+t^2)}{(1+t^2)^2}\right) \\ &= (0, 2a)\end{aligned}$$

4 do Carmo pg. 11 exercise 10

(a)

Left Equation For $\alpha(t) = (x(t), y(t), z(t))$ the Fundamental Theorem of Calculus gives us that there is a $c \in I$ such that

$$(q-p)\cdot v = (q_1 - p_1)v_1 + (q_2 - p_2)v_2 + (q_3 - p_3)v_3 = x'(c)v_1 + y'(c)v_2 + z'(c)v_3 = \alpha'(c)\cdot v$$

but since v is a vector of magnitude 1, then

$$\alpha'(c)\cdot v = \int_a^b \alpha'(t)\cdot v \; dt$$

so

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v \; dt$$

Right Inequality By the definition of dot product

$$\int_a^b \alpha'(t)\cdot v \; dt = \int_a^b |\alpha'(t)||v|\cos\theta \; dt = \int_a^b |\alpha'(t)|\cos\theta \; dt$$

Now since $\cos\theta$ ranges between -1 and 1 then

$$\int_a^b |\alpha'(t)|\cos\theta \; dt \le \int_a^b |\alpha'(t)| \; dt$$

(b)

5

(a)

To show that $\alpha(t)$ is parametrized by arc lenght, we must show $|\alpha'(t)| = 1$ for all t. So proceeding that way, we have

$$\begin{aligned} x'(t) &= 1 - \frac{1}{t+1} \\ y'(t) &= \frac{\sqrt{2t}}{t+1} \\ z'(t) &= \frac{1}{t+1} \end{aligned}$$

giving us

$$\alpha'(t) = \left(1 - \frac{1}{t+1}, \frac{\sqrt{2t}}{t+1}, \frac{1}{t+1}\right) = \left(\frac{t}{t+1}, \frac{\sqrt{2t}}{t+1}, \frac{1}{t+1}\right).$$

Therefore $|\alpha'(t)|$ is

$$\sqrt{\left(\frac{t}{t+1}\right)^2 + \left(\frac{\sqrt{2t}}{t+1}\right)^2 + \left(\frac{1}{t+1}\right)^2} = \sqrt{\frac{t^2}{(t+1)^2} + \frac{2t}{(t+1)^2} + \frac{1}{(t+1)^2}} = \sqrt{\frac{(t+1)^2}{(t+1)^2}} = 1$$

Hence α is parametrized by arc length.

(b)

Curvature. The curvature, κ, is $|\alpha''(t)|$. Using what we know from the previous part,

$$\alpha''(t) = \left(\frac{1}{t+1} - \frac{t}{(t+1)^2}, \frac{1}{\sqrt{2t}(t+1)} - \frac{\sqrt{2t}}{(t+1)^2}, \frac{-1}{(t+1)^2}\right) = \left(\frac{1}{(t+1)^2}, \frac{1-t}{\sqrt{2t}(t+1)^2}, \frac{-1}{(t+1)^2}\right)$$

then

$$\begin{aligned} \kappa(t) &= \sqrt{\left(\frac{1}{(t+1)^2}\right)^2 + \left(\frac{1-t}{\sqrt{2t}(t+1)^2}\right)^2 + \left(\frac{-1}{(t+1)^2}\right)^2} = \sqrt{\frac{2}{(t+1)^4} + \frac{(1-t)^2}{2t(t+1)^4}} = \sqrt{\frac{4t+(1-t)^2}{2t(t+1)^4}} = \sqrt{\frac{(t+1)^2}{2t(t+1)^4}} \\ &= \frac{1}{\sqrt{2t}(t+1)} \end{aligned}$$

Torsion. The torsion of α, τ, is the magnitude of the derivative of the binormal vector, b, as given by the Frenet equation $b' = \tau n$. The binomial vector is

$$
\begin{aligned}
b(t) &= t(t) \wedge n(t) \\
b(t) &= \alpha'(t) \wedge (\kappa(t))^{-}1\alpha''(t) \\
b(t) &= \left(\frac{t}{t+1}, \frac{\sqrt{2t}}{t+1}, \frac{1}{t+1}\right) \wedge \sqrt{2t}(t+1)\left(\frac{1}{(t+1)^2}, \frac{1-t}{\sqrt{2t}(t+1)^2}, \frac{-1}{(t+1)^2}\right) \\
b(t) &= \left(\frac{t}{t+1}, \frac{\sqrt{2t}}{t+1}, \frac{1}{t+1}\right) \wedge \left(\frac{\sqrt{2t}}{t+1}, \frac{1-t}{t+1}, \frac{-\sqrt{2t}}{t+1}\right) \\
b(t) &= \frac{1}{(t+1)^2}\left[\left(t, \sqrt{2t}, 1\right) \wedge \left(\sqrt{2t}, 1-t, -\sqrt{2t}\right)\right] \\
b(t) &= \frac{1}{(t+1)^2}\left(3t-1, (t-1)\sqrt{2t}, -t^2-t\right)
\end{aligned}
$$

so

$$
\begin{aligned}
b'(t) &= \frac{-2}{(t+1)^3}\left(3t-1, (t-1)\sqrt{2t}, -t^2-t\right) + \frac{1}{(t+1)^2}\left(3, \frac{3t-1}{\sqrt{2t}}, -2t-1\right) \\
b'(t) &= \frac{1}{(t+1)^3}\left(2-6t, (2-2t)\sqrt{2t}, 2t^2+2t\right) + \frac{1}{(t+1)^3}\left(3t+3, \frac{3t^2-2t-1}{\sqrt{2t}}, -2t^2-3t-1\right) \\
b'(t) &= \frac{1}{(t+1)^3}\left(-3t-5, \frac{-(t-1)^2}{\sqrt{2t}}, t-1\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\tau(t) &= |b'(t)| \\
\tau(t) &= \frac{1}{(t+1)^3}\sqrt{(-3t-5)^2 + \left(\frac{-(t-1)^2}{\sqrt{2t}}\right)^2 + (t-1)^2} \\
\tau(t) &= \frac{1}{(t+1)^3}\sqrt{10t^2+28t+26+\frac{(t-1)^4}{2t}}
\end{aligned}
$$

6 do Carmo pg. 22 exercise 1

The following are for

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right)$$

(a) The parameter s is the arc length

$$\alpha'(s) = \left(\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)$$

$$|\alpha'(s)| = \sqrt{\frac{a^2}{c^2}\sin^2\frac{s}{c} + \frac{a^2}{c^2}\cos^2\frac{s}{c} + \frac{b^2}{c^2}} = \sqrt{\frac{a^2}{c^2}+\frac{b^2}{c^2}} = \sqrt{\frac{a^2+b^2}{c^2}} = \sqrt{\frac{c^2}{c^2}} = 1$$

Therefore s is the arc length.

(b) Curvature and torsion.

$$\alpha''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)$$

$$k(s) = |\alpha''(s)| = \sqrt{\left(\frac{a}{c^2}\cos\frac{s}{c}\right)^2 + \left(\frac{a}{c^2}\sin\frac{s}{c}\right)^2} = \frac{a}{c^2}$$

$$\begin{aligned}
b(s) &= \alpha'(s) \wedge (k(s)^{-1})\alpha''(s) \\
&= \left(\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \wedge \frac{c^2}{a}\left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right) \\
&= \left(\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \wedge \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right) \\
&= \left(\frac{b}{c}\sin\frac{s}{c}, \frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\sin^2\frac{s}{c} + \frac{a}{c}\cos^2\frac{s}{c}\right) \\
&= \left(\frac{b}{c}\sin\frac{s}{c}, \frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right)
\end{aligned}$$

$$b'(s) = \left(\frac{b}{c^2}\cos\frac{s}{c}, -\frac{b}{c^2}\sin\frac{s}{c}, 0\right)$$

$$\tau(s) = |b'(s)| = \sqrt{\left(\frac{b}{c^2}\cos\frac{s}{c}\right)^2 + \left(\frac{b}{c^2}\sin\frac{s}{c}\right)^2} = \sqrt{\frac{b^2}{c^4}\cos^2\frac{s}{c} + \frac{b^2}{c^4}\sin^2\frac{s}{c}} = \sqrt{\frac{b^2}{c^4}} = \frac{b}{c^2}$$

(c)

The osculating plane of α at $s = s_0$ is defined by the following point, normal vector tuple

$$(\alpha(s_0), b(s_0)) = \left(\left(a\cos\frac{s_0}{c}, a\sin\frac{s_0}{c}, b\frac{s_0}{c}\right), \left(\frac{b}{c}\sin\frac{s_0}{c}, \frac{b}{c}\cos\frac{s_0}{c}, \frac{a}{c}\right)\right)$$

the equations for which are given and determined in the previous problem.

(d)

The angle between the z-axis and a line passing through $\alpha(s)$ and containing $n(s)$ will be the same as the angle between $n(s)$ and any vector contain by the z-axis, say $(0, 0, 1)$. And that angle will be the inverse cosine of their dot product, since both vectors have magnitude of one.

Since $n(s)(k(s))^{-1}\alpha''(s) = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$, then the dot product of it and $(0, 0, 1)$ is zero, implying that the cosine of the angle between them is zero, which in turn confirms that the angle between them is $\frac{\pi}{2}$.

(e)

This is similar to the previous problem except that we will use $t(s)$ in place of $n(s)$. Because

$$t(s) \cdot (0, 0, 1) = \left(\frac{-a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) \cdot (0, 0, 1) = \frac{b}{c}$$

i.e. the cosine of the angle between them is a constant, then the angle bewteen t and $(0, 0, 1)$ is constant, since they both have a magnitude of one.

7 do Carmo pg. 25 exercise 12

(a)

Since

$$s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$$

then by the fundamental theorem of calculus we have $ds/dt = |\alpha'(t)|$, which implies $dt/ds = 1/|\alpha'(t)|$.

Taking the derivative of both sides of the above equation leads to the following sequence of equations.

$$\begin{aligned}
d^2t/d^2s &= \frac{d}{ds}\frac{1}{|\alpha'(t)|} \\
&= \frac{d}{ds}\frac{1}{\sqrt{\alpha' \cdot \alpha'}} \\
&= \frac{-\left(\alpha'' \cdot \alpha' \frac{dt}{ds} + \alpha' \cdot \alpha'' \frac{dt}{ds}\right)}{2\sqrt{(\alpha' \cdot \alpha')^3}} \\
&= \frac{-(\alpha'' \cdot \alpha') \frac{dt}{ds}}{|\alpha'|^3} \\
&= \frac{-(\alpha'' \cdot \alpha')}{|\alpha'|^4}
\end{aligned}$$

(b)

(c)

(d)

Assuming that the yet-to-be-proven part (b) of this problem is true, then the signed curvature is

$$\begin{aligned}
k(t) &= \frac{\alpha' \wedge \alpha''}{|\alpha'|^3} \\
&= \frac{(x', y') \wedge (x'', y''}{|(x', y')|^3} \\
&= \frac{x'y'' - x''y'}{\left(\sqrt{(x')^2 + (y')^2}\right)^3} \\
&= \frac{x'y'' - x''y'}{\left((x')^2 + (y')^2\right)^{3/2}}
\end{aligned}$$