Math 501: Differential Geometry
Homework 7

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http://coursework.tylerlogic.com/courses/math501/homework07

1

Taken off the homework.

2

3 do Carmo pg 212 problem 11

Let X be a parametrization of a surface with normal N. Define Y to be

$$Y(u,v) = X(u,v) + aN(u,v) \tag{3.1}$$

for some positive a.

(a)

We know the following to hold

$$\begin{aligned} N_u &= a_{11}X_u + a_{21}X_v \\ N_v &= a_{12}X_u + a_{22}X_v \end{aligned}$$

where (a_{ij}) is the matrix representation of the differential of N. Equation 3.1 yields

$$\begin{aligned} Y_u &= X_u + aN_u \\ Y_v &= X_v + aN_v \end{aligned}$$

and thus combining the two sets of equations above we are left with

$$\begin{aligned} Y_u &= X_u + a(a_{11}X_u + a_{21}X_v) = (1 + aa_{11})X_u + aa_{21}X_v \\ Y_v &= X_v + a(a_{12}X_u + a_{22}X_v) = aa_{12}X_u + (1 + aa_{22})X_v \end{aligned}$$

With this, we can take the cross product of Y_u and Y_v revealing that

$$\begin{aligned} Y_u \wedge Y_v &= ((1 + aa_{11})X_u + aa_{21}X_v) \wedge (aa_{12}X_u + (1 + aa_{22})X_v) \\ &= (1 + aa_{11})aa_{12}X_u \wedge X_u + aa_{21}aa_{12}X_v \wedge X_u + (1 + aa_{11})(1 + aa22)X_u \wedge X_v + aa_{21}(1 + aa_{22})X_v \wedge X_v \\ &= aa_{21}aa_{12}X_v \wedge X_u + (1 + aa_{11})(1 + aa_{22})X_u \wedge X_v \\ &= -aa_{21}aa_{12}X_u \wedge X_v + (1 + aa_{11})(1 + aa_{22})X_u \wedge X_v \\ &= -aa_{21}aa_{12}X_u \wedge X_v + (1 + aa_{11} + aa_{22} + aa_{11}aa_{22})X_u \wedge X_v \\ &= (1 + a(a_{11} + a_{22}) + a^2(a_{11}a_{22} - a_{21}a_{12}))X_u \wedge X_v \end{aligned}$$

Now since $K = \det([dN])$ and $H = -1/2\,\mathrm{tr}([dN])$ for Gaussian and mean curvatures of X, then

$$a_{11} + a_{22} = -2H$$

and

$$a_{11}a_{22} - a_{21}a_{12} = K$$

resulting in

$$Y_u \wedge Y_v = (1 - 2Ha + Ka^2)X_u \wedge X_v$$

(b)

Let F be the homeomorphism from S to the parallel surface defined by $F(p) = p + aN(p)$. Thus we have

$$\begin{aligned} F_u &= X_u + aN_u = X_u + a(a_{11}X_u + a_{21}X_v) = (1 + aa_{11})X_u + aa_{21}X_v \\ F_v &= X_v + aN_v = X_v + a(a_{12}X_u + a_{22}X_v) = aa_{12}X_u + (1 + aa_{22})X_v \end{aligned}$$

indicating that

$$[dF_p]_{\{X_u,X_v\}} = \Big(dF_p(X_u)\ dF_p(X_v)\Big) = \Big(F_u\ F_v\Big) = \Big([F_u]_X\ [F_v]_X\Big) = \begin{pmatrix} 1 + aa_{11} & aa_{12} \\ aa_{21} & 1 + aa_{22} \end{pmatrix}$$

which results in

$$\det(dF_p) = 1 + aa_{11} + aa_{22} + aa_{11}aa_{22} - aa_{12}aa_{21} = 1 + a(a_{11} + a_{22}) + a^2(a_{11}a_{22} - a_{12}a_{21}) = 1 - 2Ha + Ka^2 \quad (3.2)$$

Now because of part (a), we know that the normal field for the parallel surface, call it M, at $F(p)$ is the same as the normal field for S at p, i.e.

$$N(p) = M(F(p)) \quad (3.3)$$

We will make use of this to determine the Gaussian and mean curvatures.

Gaussian Curvature Using the chain rule with Equation 3.3 gives us that

$$\begin{aligned} dN_p &= dM_{F(p)}dF_p \\ \det(dN_p) &= \det\left(dM_{F(p)}dF_p\right) \\ \det(dN_p) &= \det\left(dM_{F(p)}\right)\det(dF_p) \end{aligned}$$

which leads to

$$\det\left(dM_{F(p)}\right) = \frac{\det(dN_p)}{\det(dF_p)} = \frac{K}{\det(dF_p)} \quad (3.4)$$

The combination of Equation 3.2 and Equation 3.4 leads to the parallel surface having a Gaussian curvature, $\overline{K}$, of

$$\overline{K} = \det(dM_{F(p)}) = \frac{K}{1 - 2Ha + Ka^2}$$

Mean Curvature Again using the chain rule with Equation 3.3 and employing Lemma A.1 and Equation 3.4 we find that

$$\begin{aligned}
dN_p &= dM_{F(p)} dF_p \\
dN_p^{-1} &= dF_p^{-1} dM_{F(p)}^{-1} \\
dF_p dN_p^{-1} &= dM_{F(p)}^{-1} \\
\operatorname{tr}(dM_{F(p)}^{-1}) &= \operatorname{tr}(dF_p dN_p^{-1}) \\
\frac{\operatorname{tr}(dM_{F(p)})}{\det(dM_{F(p)}} &= \operatorname{tr}([dF_p]_{\{X_u,X_v\}}[dN_p]_{\{X_u,X_v\}}^{-1}) \\
\operatorname{tr}(dM_{F(p)}) &= \det(dM_{F(p)}) \operatorname{tr}\left(\begin{pmatrix} 1+aa_{11} & aa_{12} \\ aa_{21} & 1+aa_{22} \end{pmatrix} \frac{1}{\det(dN_p)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right) \\
\operatorname{tr}(dM_{F(p)}) &= \frac{\det(dN_p)}{\det(dF_p)} \frac{1}{\det(dN_p)} \operatorname{tr}\left(\begin{pmatrix} 1+aa_{11} & aa_{12} \\ aa_{21} & 1+aa_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}\right) \\
\operatorname{tr}(dM_{F(p)}) &= \frac{1}{1-2Ha+Ka^2}(a_{22}+aa_{11}a_{22}-aa_{12}a_{21}-aa_{12}a_{21}+a_{11}+aa_{11}a_{22}) \\
\operatorname{tr}(dM_{F(p)}) &= \frac{1}{1-2Ha+Ka^2}((a_{11}+a_{22})+2a(a_{11}a_{22}-a_{12}a_{21})) \\
\operatorname{tr}(dM_{F(p)}) &= \frac{1}{1-2Ha+Ka^2}(-2H+2aK) \\
\operatorname{tr}(dM_{F(p)}) &= \frac{-2(H-aK)}{1-2Ha+Ka^2}
\end{aligned}$$

Hence the mean curvature, $\overline{H}$, is

$$\overline{H} = -\frac{1}{2}\operatorname{tr}(dM_{F(p)}) = \frac{H-aK}{1-2Ha+Ka^2}$$

(c)

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4 do Carmo pg 229 problem 9

Let S_1 and S_2 be regular surfaces with a conformal maps $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$.

(a) Inverses of isometries are isometries

The proof in Problem section 8 part (a) holds for this when λ is the constant function of 1.

(b) Composition of isometries is an isometry

The proof in Problem section 8 part (b) holds for this when λ_φ and λ_ψ are both the constant functions of 1.

5 do Carmo pg 229 problem 10

Let $\varphi : S \to S$ be a rotation about the axis of a surface of revolution, S. Because it is simply a rotation, φ is the restriction of some linear map of rotation, $R : \mathbb{R}^3 \to \mathbb{R}^3$, to S. Hence for $v \in S$, $\varphi(p) = Ap$ for some matrix

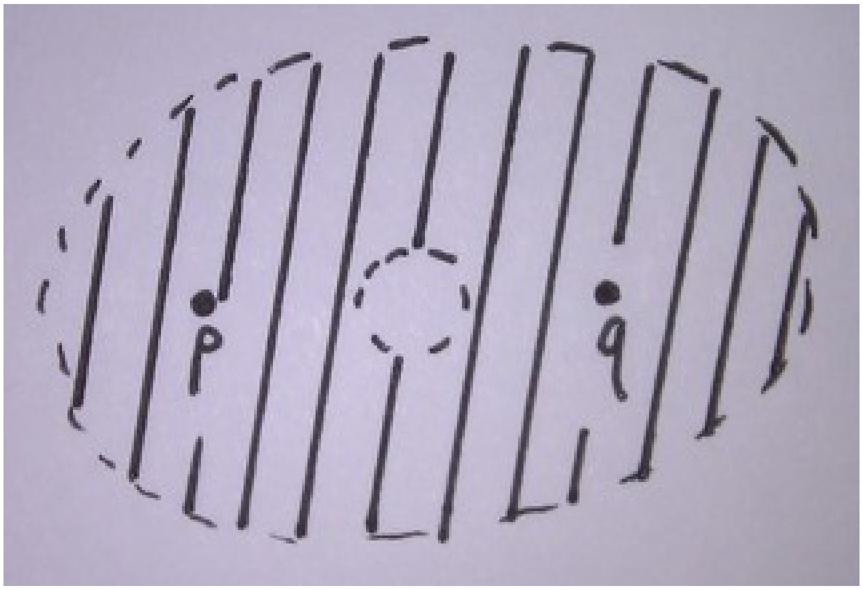

Figure 1: A *planar* surface with two points for no curve between the two has a length equal to the intrinsic distance.

of rotation A. Note that rotational matrices such as A are orthogonal. Thus we have the following for $p \in S$ and $v \in T_pS$ with some curve α such that $\alpha(0) = p$ and $\alpha'(0) = v$

$$d\varphi_p(v) = \left.\frac{d}{dt}\right|_{t=0} \varphi(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} A\alpha(t)$$

but because A is not dependent on t

$$\left.\frac{d}{dt}\right|_{t=0} A\alpha(t) = A\alpha'(t)|_{t=0} = A\alpha'(0) = Av$$

resulting in $d\varphi_p(v) = Av$. Thus since A is orthogonal then for any $v, w \in T_pS$

$$\langle d\varphi_p(v), d\varphi_p(w)\rangle = \langle Av, Aw\rangle = \langle v, w\rangle$$

thereby giving us that φ is an isometry.

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(a)

The drawing in Figure 1 has two points p and q in a *planar* surface for which there is no curve between them with length equal to the intrinsic distance between the two points. A curve that would potentially have a length of the intrinsic distance would need to go through the hole in the middle of the surface, but it obviously cannot while remaining a curve of the surface.

(b)

From our first homework assignment, we know that for any curve α in S with $\alpha(a) = p$ and $\alpha(b) = q$, $L(\alpha)_a^b \geq |p-q|$. Thus because $d(p,q)$ is the infimum a set of the lengths (from a to b) of curves in S which pass through p and q at a and b, respectively, then $d(p,q) \geq |p-q|$.

(c)

7

(a) Coefficients of the first fundamental form.

$$
\begin{aligned}
X(\phi,\theta) &= (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)\\
X_\phi &= (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)\\
X_\theta &= (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)\\
E &= \langle X_\phi, X_\phi\rangle = \cos^2\theta\sin^2\phi + \cos^2\theta\cos^2\phi = \cos^2\theta\\
F &= \langle X_\phi, X_\theta\rangle = \cos\theta\sin\phi\sin\theta\cos\phi - \cos\theta\cos\phi\sin\theta\sin\phi = 0\\
G &= \langle X_\theta, X_\theta\rangle = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1
\end{aligned}
$$

(b) Relation of M, $\tilde{M}$, X, and Y

Since $\tilde{M}$ is the map from the domain of X to the domain of Y, then we have

$$M(X(\phi,\theta) = Y(\tilde{M}(\phi,\theta)) \tag{7.5}$$

for (ϕ,θ) in the domain of X.

(c) Coefficients of the first fundamental form of $\overline{X}$

By Equation 7.5 we have

$$\overline{X}(\phi,\theta) = M(X(\phi,\theta) = Y(\tilde{M}(\phi,\theta)) = Y(\phi, z(\theta)) = (\cos\phi, \sin\phi, z(\theta))$$

which results in

$$
\begin{aligned}
\overline{X}_\phi &= (-\sin\phi, \cos\phi, 0)\\
\overline{X}_\theta &= (0, 0, z'(\theta))\\
\overline{E} &= \left\langle \overline{X}_\phi, \overline{X}_\phi \right\rangle = \sin^2\phi + \cos^2\phi = 1\\
\overline{F} &= \left\langle \overline{X}_\phi, \overline{X}_\theta \right\rangle = 0\\
\overline{G} &= \left\langle \overline{X}_\theta, \overline{X}_\theta \right\rangle = (z'(\theta))^2
\end{aligned}
$$

(d)

In order to have M be a conformal map, we must satisfy

$$
\begin{aligned}
\overline{E} &= \lambda^2 E\\
1 &= \lambda^2\cos^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\overline{F} &= \lambda^2 F\\
0 &= 0
\end{aligned}
$$

and

$$\begin{aligned}\overline{G} &= \lambda^2 G \\ (z'(\theta))^2 &= \lambda^2\end{aligned}$$

indicating that $z'(\theta)$ must be $\sec\theta$.

(e)

From the result of the previous part of the problem we know that

$$z(\theta) = \int \sec\theta = \ln|\sec\theta + \tan\theta|$$

assuming $z(0) = 0$ and $z(\theta) > 0$ for $\theta \in (0, \pi/2)$.

8

Let S_1 and S_2 be regular surfaces with a conformal maps $\varphi : S_1 \to S_2$ and $\psi : S_2 \to S_3$.

(a) Inverses of conformal maps are conformal

Since S_1 and S_2 are diffeomorphic, then $T_pS_1 = T_{\varphi(p)}S_2$ and $d\varphi_p = d\varphi^{-1}_{\varphi(p)}$ for $p \in S_2$. Thus for vectors $v_1, v_2 \in T_pS_2 = T_{\varphi^{-1}(p)}S_1$, the vectors $d\varphi_p^{-1}(v_1)$ and $d\varphi_p^{-1}(v_2)$ are vectors in $T_p(S_1)$. So we have

$$\left\langle d\varphi_{\varphi^{-1}(p)}(d\varphi_p^{-1}(v_1)), d\varphi_{\varphi^{-1}(p)}(d\varphi_p^{-1}(v_2))\right\rangle = \lambda^2(p)\left\langle d\varphi_p^{-1}(v_1), d\varphi_p^{-1}(v_2)\right\rangle$$

which in turn implies

$$\left\langle d(\varphi\circ\varphi^{-1})_p(v_1), d(\varphi\circ\varphi^{-1})_p(v_2)\right\rangle = \lambda^2(p)\left\langle d\varphi_p^{-1}(v_1), d\varphi_p^{-1}(v_2)\right\rangle$$

however, $\varphi\circ\varphi^{-1}$ is the identity map implying that the above equation simplifies to

$$\langle v_1, v_2\rangle = \lambda^2(p)\left\langle d\varphi_p^{-1}(v_1), d\varphi_p^{-1}(v_2)\right\rangle$$

which gives us what we're looking for

$$\left\langle d\varphi_p^{-1}(v_1), d\varphi_p^{-1}(v_2)\right\rangle = \frac{1}{\lambda^2(p)}\langle v_1, v_2\rangle$$

(b) Composition of conformal maps is conformal

Let $p \in S_1$ and $v_1, v_2 \in T_pS$. Then we have the following

$$\begin{aligned}\langle d(\psi\circ\varphi)_p(v_1), d(\psi\circ\varphi)_p(v_2)\rangle &= \left\langle d\psi_{\varphi(p)}(d\varphi_p(v_1)), d\psi_{\varphi(p)}(d\varphi_p(v_1))\right\rangle \\ &= \lambda_\psi^2(\varphi(p))\langle d\varphi_p(v_1), d\varphi_p(v_1)\rangle \\ &= \lambda_\psi^2(\varphi(p))\lambda_\varphi^2(p)\langle v_1, v_2\rangle\end{aligned}$$

so for $\lambda(p) = \lambda_\psi(\varphi(p))\lambda_\varphi(p)$ we have

$$\langle d(\psi\circ\varphi)_p(v_1), d(\psi\circ\varphi)_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle$$

giving us the fact that $\varphi\circ\psi$ is conformal since it is a diffeomorphism.

(c)

Appendix

A Helpful Lemmas

Lemma A.1 *The trace of the inverse of a two-dimensional matrix A is*

$$\operatorname{tr}(A^{-1}) = \frac{\operatorname{tr}(A)}{\det(A)}$$

Proof. Let A be the matrix denoted by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which in turn leads to

$$\operatorname{tr}(A^{-1}) = \operatorname{tr}\left(\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right) = \frac{1}{\det(A)} \operatorname{tr} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{d+a}{\det(A)} = \frac{\operatorname{tr}(A)}{\det(A)}$$

□