Math 502: Abstract Algebra
Homework 4

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(a) Prove that there exists a unique group homomorphism $\mathrm{sgn} : S_n \to \mu_2$

Homomorphic Let σ, τ each be elements of S_n. Then we have

$$\begin{aligned}
\mathrm{sgn}(\sigma\tau) f_n(x_1, \ldots, x_n) &= f_n(x_{(\sigma\tau)(1)}, \ldots, x_{(\sigma\tau)(1)}) \\
&= f_n(x_{\sigma(\tau(1))}, \ldots, x_{\sigma(\tau(1))}) \\
&= \mathrm{sgn}(\sigma) f_n(x_{\tau(1)}, \ldots, x_{\tau(1)}) \\
&= \mathrm{sgn}(\sigma) f_n(x_{\tau(1)}, \ldots, x_{\tau(1)}) \\
&= \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) f_n(x_1, \ldots, x_n)
\end{aligned}$$

and because $\mathbb{Q}$ is a field, we can cancel $f_n(x_1, \ldots, x_n)$ on both sides leaving us with $\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)$.

Uniqueness For later contradiction, let $g : S_n \to \mu_2$ be another group homomorphism not equal to sgn that satisfies the same properties of the sgn function. Then for $\sigma \in S_n$, we have

$$f_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = g(\sigma) f_n(x_1, \ldots, x_n)$$

but there must exist at least one τ where $\mathrm{sgn}(\tau) \neq g(\tau)$. So without loss of generality, let $\mathrm{sgn}(\tau) = 1$ and $g(\tau) = -1$. However, then we would have

$$f_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \mathrm{sgn}(\sigma) f_n(x_1, \ldots, x_n) = f_n(x_1, \ldots, x_n)$$

and

$$f_n(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = g(\sigma) f_n(x_1, \ldots, x_n) = -f_n(x_1, \ldots, x_n)$$

which is not possible with our definition of f_n. Hence we've reached a contradiction, and thus g and sgn must be one in the same.

(b)

(c)

The following proof is derived from [DF04, pg. 109]. It's just so slick.

We will first prove that the transposition $(1, 2)$ has negative sign and then move on to the general case. According to the definition of the sgn function we have that for $\sigma = (1, 2)$,

$$\begin{aligned}
f_n(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) &= \mathrm{sgn}(\sigma) f_n(x_1, x_2, \ldots, x_n) \\
\Pi_{1 \le i < j \le n}(x_{\sigma(i)} - x_{\sigma(j)}) &= \mathrm{sgn}(\sigma) \Pi_{1 \le i < j \le n}(x_i - x_j) \\
\Pi_{2 \le j \le n}(x_{\sigma(1)} - x_{\sigma(j)}) \Pi_{3 \le j \le n}(x_{\sigma(2)} - x_{\sigma(j)}) \Pi_{3 \le i < j \le n}(x_i - x_j) &= \mathrm{sgn}(\sigma) \Pi_{2 \le j \le n}(x_1 - x_j) \Pi_{3 \le j \le n}(x_2 - x_j) \\
&\quad \Pi_{3 \le i < j \le n}(x_i - x_j) \\
\Pi_{2 \le j \le n}(x_{\sigma(1)} - x_{\sigma(j)}) \Pi_{3 \le j \le n}(x_{\sigma(2)} - x_{\sigma(j)}) &= \mathrm{sgn}(\sigma) \Pi_{2 \le j \le n}(x_1 - x_j) \Pi_{3 \le j \le n}(x_2 - x_j) \\
(x_{\sigma(1)} - x_{\sigma(2)}) \Pi_{3 \le j \le n}(x_{\sigma(1)} - x_{\sigma(j)}) \Pi_{3 \le j \le n}(x_{\sigma(2)} - x_{\sigma(j)}) &= \mathrm{sgn}(\sigma)(x_1 - x_2) \Pi_{3 \le j \le n}(x_1 - x_j) \Pi_{3 \le j \le n}(x_2 - x_j) \\
(x_2 - x_1) \Pi_{3 \le j \le n}(x_2 - x_j) \Pi_{3 \le j \le n}(x_1 - x_j) &= \mathrm{sgn}(\sigma)(x_1 - x_2) \Pi_{3 \le j \le n}(x_1 - x_j) \Pi_{3 \le j \le n}(x_2 - x_j) \\
(x_2 - x_1) &= \mathrm{sgn}(\sigma)(x_1 - x_2)
\end{aligned}$$

which indicates that $\mathrm{sgn}((1\ 2)) = -1$.

Now let (i, j) be an abitrary transposition in S_n. Then we have that $(i, j) = (1, i)(2, j)(1, 2)(1, i)(2, j)$. This then yields the following

$$\begin{aligned}
\operatorname{sgn}((i,j)) &= \operatorname{sgn}((1,i)(2,j)(1,2)(1,i)(2,j)) \\
&= \operatorname{sgn}((1,i))\operatorname{sgn}((2,j))\operatorname{sgn}((1,2))\operatorname{sgn}((1,i))\operatorname{sgn}((2,j)) \\
&= \operatorname{sgn}((1,i))^2\operatorname{sgn}((2,j))^2\operatorname{sgn}((1,2)) \\
&= -1
\end{aligned}$$

where we make use of the fact that sgn is a homomorphism and the commutativity of μ_2.

(d)

We can see that a cycle of length m has that $(1, 2, \ldots, m) = (1, 2)(2, 3) \cdots (m - 1, m)$. Therefore, using the fact that the sgn function is a homomorphism and that transpositions have sign of -1, we obtain

$$\operatorname{sgn}((1,2,\ldots,m)) = \operatorname{sgn}((1,2)(2,3)\cdots(m-1,m)) = \operatorname{sgn}((1,2))\operatorname{sgn}((2,3))\cdots\operatorname{sgn}((m-1,m)) = (-1)^{m-1}$$

(e)

(f) Extra Credit

(g) Extra Credit

2

For a matrix $A \in \mathrm{GL}_3(\mathbb{R})$, left $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation of left multiplication by A.

(a)

First, we see that any vecter v in C that is on the "surface" of C, i.e. any vecter with at least one coordinate of 1, must have that for $A \in G$, $L_A(v)$ must also be on the surface of C. The reason being because if it weren't, then there would be some scalar $\alpha > 1$ with $L_A(\alpha v) \in C$, but this can't happen as it contradicts $A(C) = C$.

Not the least important of the vectors on the surface of C are e_1, e_2, e_3. Since e_1, e_2, e_3 each have a length of which is the shortest possible length of a vector on the surface of C, a length of one, then their images under L_A must have length of at least one since they must be on the surface. On the other hand, because $e_1 + e_2 + e_3$ is also on the surface of C then an increase in any one of the sizes of e_1, e_2 or e_3 under L_A will result in $L_A(e_1 + e_2 + e_3)$ being outside of C. Hence L_A will map each of e_1, e_2 and e_3 to one of $\{\pm e_1, \pm e_2, \pm e_3\}$. However, because $A \in GL_3(\mathbb{R})$, then L_A is an isomorphism and will map e_1, e_2 and e_3 to distinct elements and furthermore to a basis. Therefore L_A preserves the size and angles between e_1, e_2 and e_3, which since those vectors are a basis for $\mathbb{R}^3$ makes A orthogonal, and thus $G \subset O_3$.

Now $I \in G$ since it preserves C. If $A, B \in G$, then $A(C) = C$ and $B(C) = C$, implying that $(BA)(C) = B(A(C)) = B(C) = C$ and therefore $AB \in G$. And finally, since $A(C) = C$ then $C = A^{-1}(C)$, and thus $A^{-1} \in G$. These three things, combined with the fact $G \subset O_3$ proven above makes G a subgroup of O_3.

(b)

Let $A \in G$. Since L_A maps $\{e_1, e_2, e_3\}$ to a basis of elements of $\{\pm e_1, \pm e_2, \pm e_3\}$ as per above, then L_A maps e_1 to one of the 6 elements of $\{\pm e_1, \pm e_2, \pm e_3\}$, maps e_2 to one of the four elements of $\{\pm e_1, \pm e_2, \pm e_3\} - \{\pm L_A(e_1)\}$ and e_3 to two of elements of $\{\pm e_1, \pm e_2, \pm e_3\} - \{\pm L_A(e_1), \pm L_A(e_2)\}$. Hence G has $6(4)(2) = 48$ elements and consists of

$$\begin{aligned}
\left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\} \quad \cup \quad & \left\{ \begin{pmatrix} a & & \\ & & b \\ & c & \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\} \\
\cup \quad & \left\{ \begin{pmatrix} & a & \\ b & & \\ & & c \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\} \\
\cup \quad & \left\{ \begin{pmatrix} & a & \\ & & b \\ c & & \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\} \\
\cup \quad & \left\{ \begin{pmatrix} & & a \\ b & & \\ & c & \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\} \\
\cup \quad & \left\{ \begin{pmatrix} & & a \\ & b & \\ c & & \end{pmatrix} \middle| a, b, c \in \{1, -1\} \right\}
\end{aligned}$$

(c)

We will abuse notation here and consider S_3 and $\mathrm{Perm}(\{L_1, L_2, L_3\})$ to be one in the same. They are, after all, isomorphic.

π is surjective Let $\sigma \in S_3$ and define A to be the matrix with i^{th} column $e_{\sigma(i)}$. Therefore according to the previous part of this problem, $A \in G$. Thus $Ae_i = e_{\sigma(i)}$ which implies that $A(L_i) = L_{\sigma(i)}$ and therefore $\pi(A) = \sigma$.

Ker(π) The kernel of the π will be the set of all matrices A such that $A(L_i) = L_i$. Therefore A will need to be such that $Ae_i = \pm e_i$. Hence

$$\mathrm{Ker}(\pi) = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| a, b, c \in \{-1, 1\} \right\}$$

(d)

Denote the set of diagonals of the problem's statement by $D = \{d_1, d_2, d_3, d_4\}$. We will again abuse notation here and consider S_4 and $\mathrm{Perm}(\{d_1, d_2, d_3, d_4\})$ to be one in the same.

We will use the following for vectors to help in our discussion here.

$$\begin{aligned}
v_1 &= e_1 + e_2 + e_3 \\
v_2 &= -e_1 + e_2 + e_3 \\
v_3 &= e_1 - e_2 + e_3 \\
v_4 &= -e_1 - e_2 + e_3
\end{aligned}$$

These vectors are the vectors contained in the diagonals of D where e_3 is always positive. With these vectors, we have a basis $\mathscr{B} = \{v_1, v_3, v_4\}$ and can write the standard basis $\mathscr{E} = \{e_1, e_2, e_3\}$ as

$$\begin{aligned} e_1 &= \frac{1}{2}(v_3 - v_4) && (2.1)\\ e_2 &= \frac{1}{2}(v_1 - v_3) && (2.2)\\ e_3 &= \frac{1}{2}(v_1 + v_4) && (2.3) \end{aligned}$$

which gives rise to

$$\begin{aligned} [\mathrm{id}]_{\mathscr{B}\mathscr{E}} &= \frac{1}{2}\begin{pmatrix} & 1 & 1 \\ 1 & -1 & \\ -1 & & 1 \end{pmatrix} && (2.4)\\ [\mathrm{id}]_{\mathscr{E}\mathscr{B}} &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} && (2.5) \end{aligned}$$

where $[\mathrm{id}]_{\mathscr{B}\mathscr{E}}$ and $[\mathrm{id}]_{\mathscr{E}\mathscr{B}}$ are, respectively, the change of basis matrix from $\mathscr{E}$ to $\mathscr{B}$ and from $\mathscr{B}$ to $\mathscr{E}$ (yes it looks backwards).

ϕ is surjective Let $\sigma \in S_4$. We then aim to find a matrix $A \in G$ such that $Av_i = v_{\sigma(i)}$. So let's define a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by its transformation of the basis $\mathscr{B}$: $T(v_i) = v_{\sigma(i)}$. By equations 2.1 through 2.3 we have that

$$\begin{aligned} T(e_1) &= \frac{1}{2}(v_{\sigma(3)} - v_{\sigma(4)})\\ T(e_2) &= \frac{1}{2}(v_{\sigma(1)} - v_{\sigma(3)})\\ T(e_3) &= \frac{1}{2}(v_{\sigma(1)} + v_{\sigma(4)}) \end{aligned}$$

leaving us with the following matrix representation of T in the standard basis

$$A = \frac{1}{2}\begin{pmatrix} v_{\sigma(3)} - v_{\sigma(4)} & v_{\sigma(1)} - v_{\sigma(3)} & v_{\sigma(1)} + v_{\sigma(4)} \end{pmatrix}$$

The definition of T informs us that $A \in G$ and that $A(d_i) = d_{\sigma(i)}$, thus implying that $\phi(A) = \sigma$.

Ker(ϕ) We know that every element of the kernel will have that $d_i = d_i$, which implies that $e_1 + e_2 + e_3$, being contained in a diagonal, will need to be mapped to itself or $-e_1 - e_2 - e_3$. Hence

$$\mathrm{Ker}(\phi) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \right\}$$

(e)

Is $\det|_G = \operatorname{sgn} \circ \pi$ This is not true. According to part (c) of this problem, $A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ is in the kernel of π, meaning that $\operatorname{sgn} \circ \pi(A) = 1$. However, A has a determinant of -1.

Is $\operatorname{sgn} \circ \pi = \operatorname{sgn} \circ \phi$

Is $\det|_G = \operatorname{sgn} \circ \phi$ This is not true. According to part (d) of this problem, $A = \begin{pmatrix} --1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ is contained in the kernel of ϕ, meaning that $\operatorname{sgn}(\phi(A)) = 1$. However, A has a determinant of -1.

(f) Extra Credit

3

(a)

The dihedral group D_8 is indeed isomorphic to a subgroup of G. Since G is the group of symmtries of a cube, then all elements which, say, keep e_1 fixed will be symmetries of the unit square contained in the plane containing e_2 and e_3. Call this square S. We can use this in our definition of an injective group homomorphism $\phi : D_8 \to G$.

We can think of ϕ as the map that takes s (where $D_8 = \langle s, r \rangle$) to the symmetry that reflects S across the plane containing e_1 and e_3; i.e. take s to the matrix representation of the linear transformation defined by $e_1 \mapsto e_1$, $e_2 \mapsto -e_2$, and $e_3 \mapsto e_3$. Similarly ϕ will take r to the matrix representation of rotation about e_3 by $\pi/2$.

This high-level definition of ϕ can be made concrete by the formula

$$\phi(s^\ell r^i) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^\ell \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^i$$

for all $s^\ell r^i \in D_8$. By noticing that

$$\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^{-1} = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}$$

we can see that the following implies that ϕ is a group homomorphism

$$\begin{aligned}
\phi(s^\ell r^i)\phi(s^k r^j) &= \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^\ell \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^i \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^k \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^j \\
&= \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^{\ell+k} \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix}^i \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^j \\
&= \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^{\ell+k} \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^{-i} \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^j \\
&= \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^{\ell+k} \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^{j-i} \\
&= \phi(s^{\ell+k} r^{j-i}) \\
&= \phi(s^\ell s^k r^{-i} r^j) \\
&= \phi(s^\ell r^i s^k r^j)
\end{aligned}$$

Now assume that some element $s^\ell r^i$ is mapped by ϕ to the identity of G. Then we would have that

$$\begin{aligned}
\phi(s^\ell r^i) &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^\ell \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^i &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^i &= \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^\ell
\end{aligned}$$

which implies that $\ell = i = 0$ since

$$\begin{aligned}
\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^2 &= \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \\
\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^3 &= \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \\
\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}^4 &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}
\end{aligned}$$

shows that the order of $\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix}$ is four. Hence the kernel is trivial, implying that ϕ is injective.

The fact that $\phi : D_8 \to G$ is an injective group homomorphism implies that D_8 is isomorphic to a subgroup of G, namely $\phi(D_8)$.

(b) Extra Credit

4

(a) Show that $\dim_{\mathbb{C}}(V) = 2$ and $\{1, j\}$ is a $\mathbb{C}$-basis of V

Let $a, b, c, d \in \mathbb{R}$. Then since

$$1(a + bi) + j(c - di) = a + bi + cj - dji = a + bi + cj + dk$$

we see that $\{1, j\}$ spans V.

Not if we were to have $1(a + bi) + j(c + di) = 0$, then $a + bi + cj - dk = 0$ which implies that $a = b = c = d = 0$. Hence 1 and j are linearly independent.

With these two results, $\{1, j\}$ is a basis for V, and thus $\dim_{\mathbb{C}}(V) = 2$.

(b) Show for $x \in \mathbb{H}$ that "left multiplication by x" is an endomorphism on V.

By the following, it can be seen that "left multiplication by x" is an endomorphism, where $v, w \in V$ The distributive law on $\mathbb{H}$ is what gives us that $\alpha(x)$ is an endomorphism, i.e.

$$\alpha(x)(v + w) = x(v + w) = xv + xw = \alpha(x)v + \alpha(x)w$$

(c) Show that $\alpha : \mathbb{H} \to \operatorname{End}_{\mathbb{C}}(V)$ is a ring homomorphism.

By the following two equations, we have that α is a ring homomorphism

$$\alpha(x+y)(v) = (x+y)(v) = xv + yv = \alpha(x)(v) + \alpha(y)(v) = (\alpha(x) + \alpha(y))(v)$$

$$\alpha(xy)(v) = (xy)(v) = x(y(v)) = \alpha(x)(\alpha(y)(v)) = (\alpha(x) \circ \alpha(y))(v)$$

(d) Prove that α is injective

Let $x \in H$ be such that $\alpha(x) = \mathrm{id}_V$. Then for all $v \in V$ $xv = \alpha(x)(v) = \mathrm{id}_V(v) = v$, but the only element of $\mathbb{H}$ with this property is the multiplicative identity, so x must be the identity. Hence the kernel of α is trivial and thus α is injective.

(e) Write down the matrix representation of $\alpha(i)$, $\alpha(j)$, and $\alpha(k)$ with respect to the basis $\{1, j\}$

Set $\mathscr{B}$ to the basis $\{1, j\}$. We will use the notation $[\cdot]_{\mathscr{B}}$ to denote "representation in the basis $\mathscr{B}$". With that said

$$\begin{aligned}
[\alpha(i)]_{\mathscr{B}} &= \begin{pmatrix} [\alpha(i)(1)]_{\mathscr{B}} & [\alpha(i)(j)]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} [i]_{\mathscr{B}} & [ij]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\
[\alpha(j)]_{\mathscr{B}} &= \begin{pmatrix} [\alpha(j)(1)]_{\mathscr{B}} & [\alpha(j)(j)]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} [j]_{\mathscr{B}} & [j^2]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} [j]_{\mathscr{B}} & [-1]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
[\alpha(k)]_{\mathscr{B}} &= \begin{pmatrix} [\alpha(k)(1)]_{\mathscr{B}} & [\alpha(k)(j)]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} [k]_{\mathscr{B}} & [kj]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} [k]_{\mathscr{B}} & [-i]_{\mathscr{B}} \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\end{aligned}$$

(f)

Because $\mathbb{H}$ is a $\mathbb{R}$-vector space of dimension four ($\{1, i, j, k\}$ is an $\mathbb{R}$-basis) and because α is injective as per part (d) of this problem, then as a linear transformation, α has a trivial kernel. Therefore the `rank+nullity` theorem informs us that the image of α has dimension of four since $\dim_{\mathbb{R}}(\mathbb{H}) = 4$. However, because $\dim_{\mathbb{C}}(\operatorname{End}_{\mathbb{C}}(V)) = 4$ as well, we must have $\alpha(\mathbb{H}) = \operatorname{End}_{\mathbb{C}}(V)$.

References

[DF04] D.S. Dummit and R.M. Foote. *Abstract Algebra*. John Wiley & Sons Canada, Limited, 2004.